

AD-A242 674



Defense Nuclear Agency
Alexandria, VA 22310-3398



DNA-TR-91-38

Axisymmetric Compression of a Mohr-Coulomb Medium with Arbitrary Dilatancy

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November 1991

Technical Report



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1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE 911101		3. REPORT TYPE AND DATES COVERED Technical 870914-901231	
4. TITLE AND SUBTITLE Axisymmetric Compression of a Mohr-Coulomb Medium with Arbitrary Dilatancy				5. FUNDING NUMBERS C-DNA 001-87-C-0259 PE-62715H PR-RS TA-RH WU-DH043510	
6. AUTHOR(S) Gary L. Wintergerst, Danny N. Burgess, and Joshua L. Merritt (BDM); Paul E. Senseny (DNA)					
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) BDM Engineering Services Company Redlands Operations P.O. Box 1206 Redlands, CA 92373-0401				8. PERFORMING ORGANIZATION REPORT NUMBER BDM/RED 87-101-01	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) Defense Nuclear Agency 6801 Telegraph Road Alexandria, VA 22310-3398 SPSD/Senseny				10. SPONSORING/MONITORING AGENCY REPORT NUMBER DNA-TR-91-38	
11. SUPPLEMENTARY NOTES This work was sponsored by the Defense Nuclear Agency under RDT&E RMC Code B4662D RS RH 00059 SPSP 3740A 25904D.					
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.				12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) A closed-form analytical solution is developed for stresses and strains in a Mohr-Coulomb medium surrounding a circular hole and loaded axisymmetrically in plane strain. This solution extends multiple-plastic-zone solutions developed by others to allow arbitrary dilatancy. Step-by-step procedures for applying the solution and numerical examples are presented.					
14. SUBJECT TERMS Tunnels Mohr-Coulomb Medium Hardening Arbitrary Dilatancy Survivability				15. NUMBER OF PAGES 110	
				16. PRICE CODE	
17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT SAR		

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

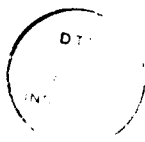
CLASSIFIED BY:

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DECLASSIFY ON:

N/A since Unclassified.

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Justification	
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Distribution	
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SECTION 1

INTRODUCTION

In this report, we develop a solution to the axisymmetric problem of a tunnel in a Mohr-Coulomb medium with internal and far-field applied pressures. The solution presented herein extends existing approaches by addressing generalized stress conditions and incorporating an arbitrary dilatancy parameter. This parameter may be varied between certain limits, but is assumed to be a constant for any given problem.

1.1 BACKGROUND.

The basic approach to this problem was defined by Newmark (Reference 1) prior to the widespread availability of high-speed digital computers and in response to the need for a method for designing underground structures to resist the effects of nuclear weapons. In the development of his solution, Newmark assumed an elastic-plastic medium with volume constancy in a plane strain configuration. Newmark's work was expanded by Hendron and Aiyer (Reference 2), who provided the solution for a dilatant material which obeys an associated flow rule. Newmark, as did Hendron and Aiyer, assumed that the out-of-plane stress (σ_z) was the intermediate principal stress and that the circumferential stress was the minimum (most compressive) principal stress. The out-of-plane stress assumption is valid for a limited range of material properties and loads; however, it is reasonable for many tunnel situations and yields a relatively simple solution for the stress and displacement fields because only one plastic zone develops.

Florence and Schwer (References 3 and 4) generalized the Hendron and Aiyer solution by eliminating the requirement that σ_z be the intermediate principal stress. Their objective was to provide an analytical solution for use in validating finite element calculations in support of experimental and theoretical investigations of buried cylinders.

Eliminating the requirement that σ_z be the intermediate principal stress substantially increases the complexity of the solution by requiring one to deal with multiple yield conditions in a single problem. As stated in Reference 3, these "depend on the relative magnitudes of the principal stresses, and these relative magnitudes depend on the values of Poisson's ratio and the friction angle." Florence and Schwer developed the solutions for two different cases. For Case I (Reference 3), there are three

different plastic zones (each with a different set of governing equations) in the material surrounding the tunnel. The problem geometry is illustrated in Figure 1 (from Reference 3). For Case II (Reference 4), there are two different plastic zones (each with a different set of governing equations) in the material surrounding the tunnel. Case II geometry is shown in Figure 2.

Detournay, St. John, and Van Dillen (Reference 5) generalized the Hendron and Aiyer solution to allow the use of an arbitrary dilatancy parameter but their solution maintained the requirement that σ_z be the intermediate principal stress. A similar solution was developed independently by Merkle (Reference 6), although we have not found it in published form. Use of the arbitrary dilatancy parameter allows the material to undergo a volume change that is intermediate between the volume constancy imposed by the Newmark solution and the strictly associated flow rule dilatancy imposed by Hendron and Aiyer. Detournay and St. John (Reference 7) assert that the dilatancy parameter should be a variable that is a function of accumulated plastic shear strain (and possibly of mean pressure). However, the use of such a variable would require the application of numerical integration in obtaining a solution.

1.2 SCOPE AND ORGANIZATION OF THE REPORT.

The solutions discussed in the previous section are "exact" only for axisymmetrically applied static loads. Nevertheless, these methods (particularly those of Newmark and of Hendron and Aiyer) have proven very useful in investigating the effects of nuclear weapons on underground structures constructed in rock. They have been used extensively by members of the DNA community for developing preliminary designs of tunnel support systems, preliminary hardness assessments of existing structures, and estimates of targeting requirements.

Because they may be programmed on a personal computer and then easily and rapidly applied, we expect these "exact" solutions will continue to be widely used in the future to obtain preliminary structural response estimates prior to performing detailed finite element calculations. Consequently, we have extended the "exact" solution one step further by combining the generalized stress state of References 3 and 4 with the concept of arbitrary dilatancy from References 5, 6, and 7.

The solution is presented in the format used by Florence and Schwer in References 3 and 4, insofar as possible. We have used the same notation, and

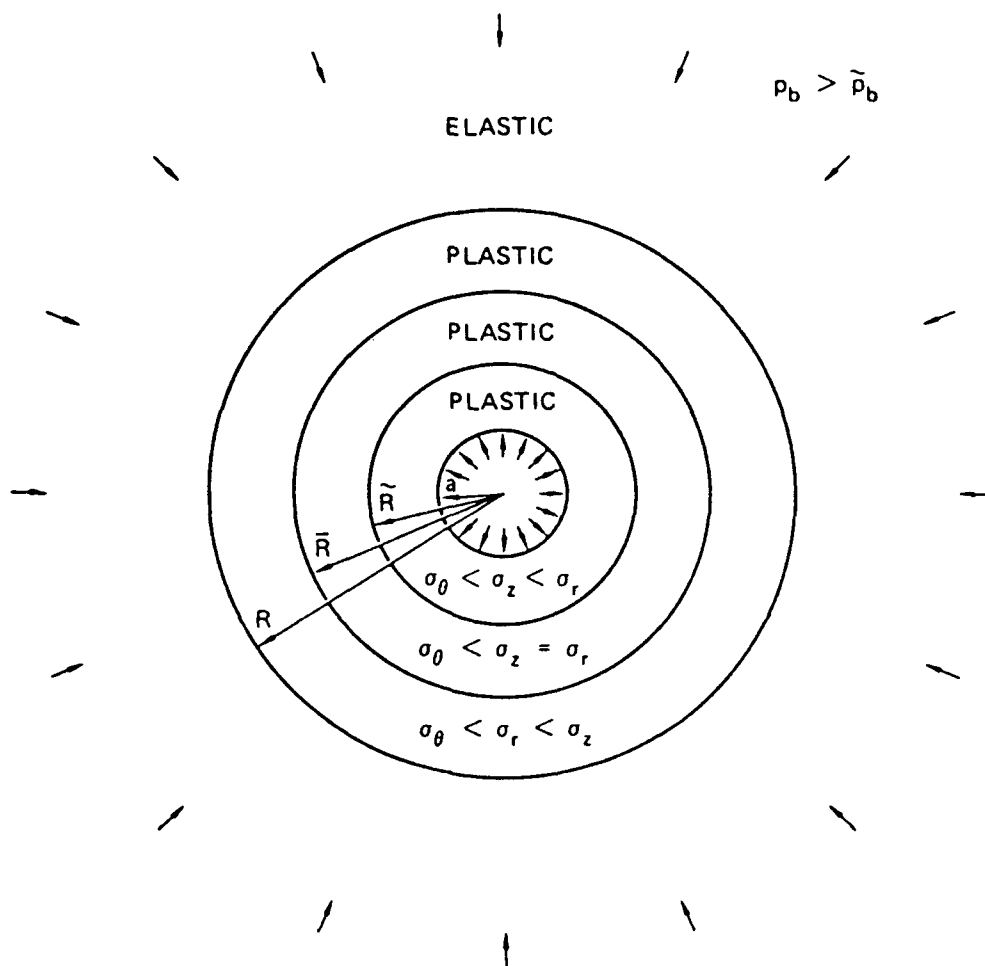


Figure 1. Outer elastic and three plastic zones (Source: Reference 3).

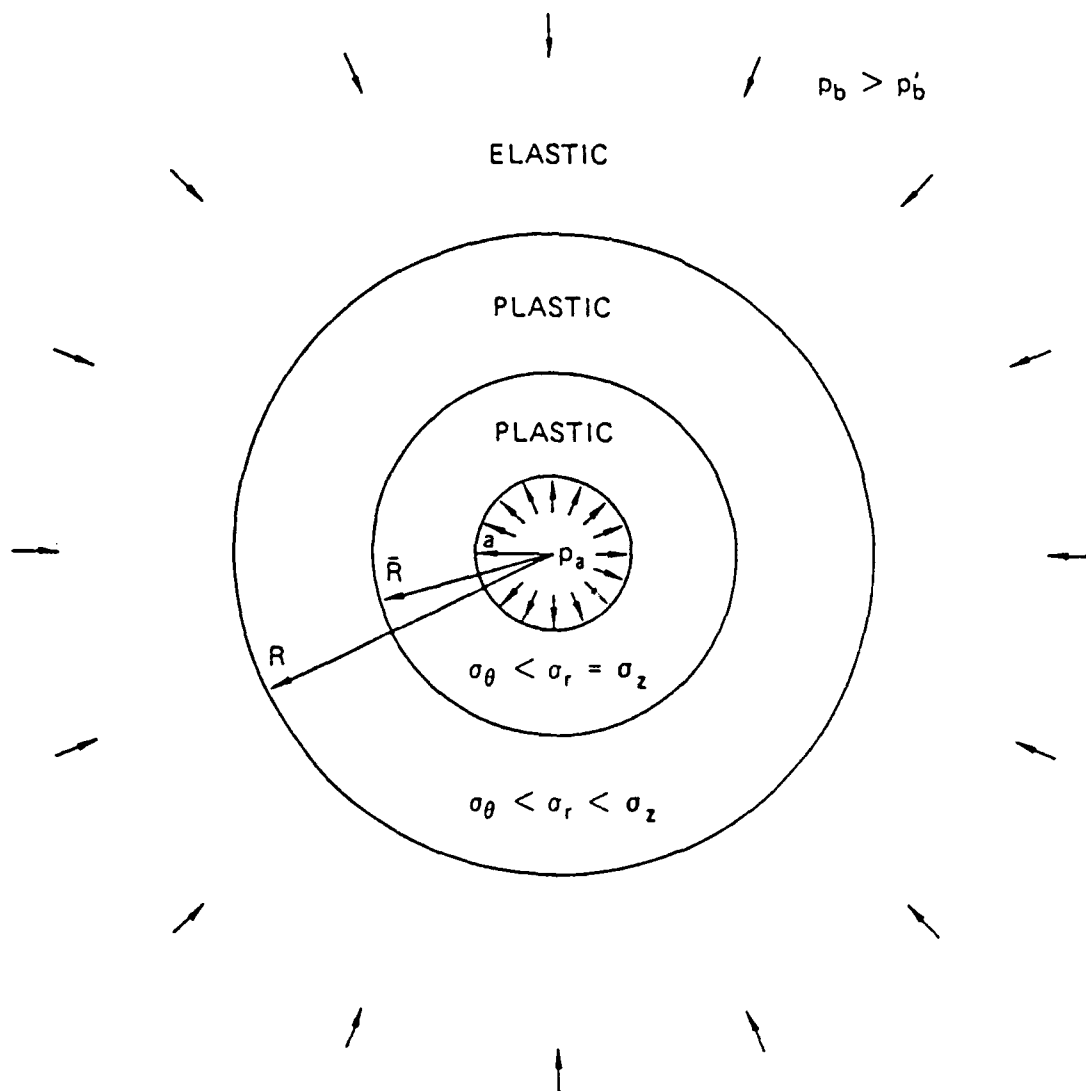


Figure 2. Outer elastic and two plastic zones (Source: Reference 4).

equations are presented in the same formats (and, to the extent possible, with the same numbers) as used by Florence and Schwer. Many of our equations are identical to theirs and much of the introductory material is largely a review of material in Reference 3. Each symbol is defined where first used, and definitions of the more frequently used symbols are summarized in Table 1. We have adopted the Florence and Schwer sign convention by defining tensile stresses and strains, applied pressures, and unconfined compressive strengths as positive. This approach allows the reader to compare equations with and without arbitrary dilatancy throughout the derivation.

In general, our approach is to replace the friction parameter "N" (defined in Equation 2 below) by the arbitrary dilatancy parameter "M" in expressions involving plastic strains or displacements. It can be demonstrated that for associated flow ($M = N$), the present solution is identical to that of References 3 and 4.

The general solution for initial yield and the conditional inequalities leading to Cases I and II are presented in Section 2. The solutions for Cases I and II are developed in Sections 3 and 4, respectively. Step-by-step procedures for applying the closed-form solutions and example problems that demonstrate their use are provided in Section 5. With the permission of the publisher, Reference 3 is presented in its entirety in Appendix A. It is our understanding that the Case II solution of Reference 4 was developed under a DNA contract, but was not published. That reference is provided in Appendix B with the concurrence of the senior author.

Table 1. Definitions of frequently used symbols.

a	- interior radius of the opening
E	- Young's modulus for medium
G	- shear modulus for medium
M	- arbitrary dilatancy parameter
N	- friction parameter
p_a	- internal pressure acting on hole boundary
p_a^*	- negative of the radial stress at the elastic-plastic boundary
p_b	- far-field pressure (compressive stress) at large radius
\bar{p}_b	- far-field pressure at initial yield
\tilde{p}_b	- far-field pressure at which $R = \tilde{R}$
p_b'	- far-field pressure when the inner plastic zone begins to form under Case II yield conditions
r	- arbitrary radius to any point in the medium
R	- radius to the elastic-plastic boundary
\bar{R}	- maximum radius at which the radial and out-of-plane stresses are equal
\tilde{R}	- minimum radius at which the radial and out-of-plane stresses are equal
R'	- radius to the elastic-plastic boundary when $p_b = p_b'$
σ_u	- unconfined compressive strength of medium
σ_r	- stress in radial direction
σ_θ	- stress in circumferential direction
σ_z	- stress in out-of-plane direction
ϵ_r	- strain in radial direction
ϵ_θ	- strain in circumferential direction
ϵ_z	- strain in out-of-plane direction
Note:	Strains are further distinguished by the superscripts (e) to denote elastic component and (p) to denote plastic component
ν	- Poisson's ratio for medium ($0 < \nu < 0.5$)
φ	- friction angle for medium
λ	- flow constant of proportionality

SECTION 2

GENERAL CONDITIONS

In this section, the general solution is developed for initial yielding of the medium around the tunnel. The conditional inequalities that lead to Cases I and II are identified. All equations in this section are identical to those in the corresponding portion of Reference 3.

2.1 INITIAL LOADING--OUTER ELASTIC ZONE.

Initially, let the internal and far-field (theoretically at infinity) pressures p_a and p_b be equal. If these pressures are increased simultaneously until the material yields, initial yielding will be governed by the yield function

$$f = \sigma_\theta - N\sigma_z + \sigma_u = 0 \quad (1)$$

where

$$N = \frac{1 + \sin\phi}{1 - \sin\phi} \quad (2)$$

Substitution of stresses ($\sigma_r = \sigma_\theta = -p_b$, $\sigma_z = -2\nu p_b$) in Equation 1 gives

$$p_a = p_b = \frac{\sigma_u}{1 - 2N\nu} \quad (3)$$

And if Equation 3 is satisfied, the material will yield completely if $2N\nu < 1$ (such yielding is precluded if $2N\nu > 1$); consequently, to insure elastic conditions in the far-field, the internal pressure will be limited such that

$$p_a < \frac{\sigma_u}{1 - 2N\nu} \quad \text{when } 2N\nu < 1 \quad (4)$$

though no such limitation is required if $2N\nu > 1$.

For the more general condition, once loading with $p_a = p_b$ reaches the desired value of p_a (e.g., the yield strength of a tunnel liner or the crush strength of backpacking material), p_b can be increased to some higher value, $p_b > p_a$. In the elastic condition, stresses and strains are given by

$$\sigma_r = -p_b + (p_b - p_a)\frac{a^2}{r^2}, \quad \sigma_\theta = -p_b - (p_b - p_a)\frac{a^2}{r^2}, \quad \sigma_z = -2\nu p_b \quad (5)$$

$$2G\epsilon_r = -(1-2\nu)p_b + (p_b - p_a)\frac{a^2}{r^2}, \quad 2G\epsilon_\theta = -(1-2\nu)p_b - (p_b - p_a)\frac{a^2}{r^2}, \quad \epsilon_z = 0 \quad (6)$$

Note that the negative sign in the third of Equations 5 was omitted in Reference 3.

2.2 INITIAL YIELDING.

Yielding will always begin at the inner surface, that is, at $r = a$. The far-field pressure associated with initial yield is defined as \bar{p}_b . In Equations 5, $\sigma_r > -p_b$, $\sigma_\theta < -p_b$, and $\sigma_z > -p_b$. Thus, the minimum (most compressive) principal stress will always be in the circumferential direction, but the maximum principal stress may be either radial or out-of-plane. If the internal pressure (and hence the radial stress at the opening) is lower in magnitude than $2\nu\bar{p}_b$ (the out-of-plane stress), then $\sigma_\theta < \sigma_z < \sigma_r$ at the onset of yield (i.e., σ_r is the maximum principal stress), so the corresponding yield function is

$$f = \sigma_\theta - N\sigma_r + \sigma_u = 0 \quad (7)$$

The radial and circumferential stresses at the interior are computed at yield from Equations 5. Substitution of those expressions in Equation 7 gives a value for the external pressure \bar{p}_b at the onset of yield.

$$\bar{p}_b = \frac{1}{2}[(N+1)p_a + \sigma_u] \quad (8)$$

In order to satisfy the inequality $p_a < 2\nu\bar{p}_b$, it is required that

$$p_a < \frac{v\sigma_u}{1-(N+1)v} \quad (N+1)v < 1 \quad (9)$$

Alternatively, where Equation 1 governs the yield condition, the yield pressure is given by

$$\bar{p}_b = \frac{p_a + \sigma_u}{2(1-Nv)} \quad Nv < 1 \quad (10)$$

For $Nv > 1$, the yield condition of Equation 7 rather than that of Equation 1 applies; consequently, Equation 8 is applicable. With Equation 10, the fol-

lowing restrictions on p_a are required

$$\frac{v\sigma_u}{1-(N+1)v} < p_a < \frac{\sigma_u}{1-2Nv} \quad 0 < Nv < \frac{1}{2} \quad (11)$$

$$\frac{v\sigma_u}{1-(N+1)v} < p_a \quad \frac{1}{2} < Nv, \quad (N+1)v < 1 \quad (12)$$

Governing conditions for the above situations are summarized in Table 2, which is equivalent to Table 1 in both References 3 and 4.

Table 2. Conditions at initial yield.

Yielding Case	Stress Order at $r = a$	Property Relations	Internal Pressure	Far-field Pressure at Initial Yield
Case I				
$p_a < 2v\bar{p}_b$	$\sigma_\theta < \sigma_z < \sigma_r$	$(N+1)v < 1$	$p_a < \frac{v\sigma_u}{1-(N+1)v}$	$\bar{p}_b = \frac{1}{2}[(N+1)p_a + \sigma_u]$
		or $1 < Nv$ $Nv < 1 < (N+1)v$	$p_a > 0$	
Case II				
$2v\bar{p}_b < p_a < \bar{p}_b$	$\sigma_\theta < \sigma_r < \sigma_z$	$\frac{1}{2} < Nv, (N+1)v < 1$	$p_a > \frac{v\sigma_u}{1-(N+1)v}$	$\bar{p}_b = \frac{p_a + \sigma_u}{2(1-Nv)}$
		$0 < Nv < \frac{1}{2}$	$\frac{v\sigma_u}{1-(N+1)v} < p_a < \frac{\sigma_u}{1-2Nv}$	

SECTION 3

CASE I SOLUTION

As stated earlier, in Case I, three separate plastic zones (with differing sets of governing equations) form in the medium surrounding the tunnel. The governing equations are developed in the four subsections that follow. In Section 3.1, the equations associated with initial growth of the plastic zone (resulting from increasing p_b) are developed. This is followed by three subsections in which the conditions governing the formation of the outer, intermediate, and inner plastic zones are developed.

3.1 INITIAL GROWTH OF THE PLASTIC ZONE.

For Case I, the yield condition of Equation 7 controls in the outer plastic zone. From equilibrium considerations we write

$$r \frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = 0 \quad (13)$$

Circumferential stress can be determined from Equation 7 and substituted in Equation 13. The result is then solved such that the radial stress at the tunnel surface ($r = a$) is equal to $-p_a$

$$\sigma_r = - \left[p_a + \frac{\sigma_u}{N-1} \right] \left(\frac{r}{a} \right)^{N-1} + \frac{\sigma_u}{N-1} \quad a < r < R \quad (14)$$

The flow rule for the yield condition of Equation 7 is given by

$$\dot{\epsilon}_r^{(p)} = \lambda \frac{\partial f}{\partial \sigma_r} = -\lambda M, \quad \dot{\epsilon}_\theta^{(p)} = \lambda \frac{\partial f}{\partial \sigma_\theta} = \lambda, \quad \dot{\epsilon}_z^{(p)} = \lambda \frac{\partial f}{\partial \sigma_z} = 0 \quad (15)$$

Note that the first relationship in Equations 15 constitutes the first departure from the derivation of Reference 3. It differs from the corresponding equation in Reference 3 only in the substitution of M for N in the radial strain component. The arbitrary dilatancy factor M , introduced in References 5 and 6, can assume any value between one and N . Dots are used here to designate differentiation with respect to external pressure. From this flow rule, the following strain equations may be written (g_1 and g_2 are unspecified functions of r).

$$\epsilon_r^{(p)} + M \epsilon_\theta^{(p)} = 0, \quad \epsilon_r^{(p)} + M \epsilon_\theta^{(p)} = g_1(r), \quad \epsilon_z^{(p)} = g_2(r) \quad (16)$$

At the elastic-plastic boundary ($r = R$), there can be no plastic strains, so we can write

$$\epsilon_r^{(p)} + M\epsilon_\theta^{(p)} = 0, \quad \epsilon_z^{(p)} = 0 \quad (17)$$

Since strain increments are the sums of their elastic and plastic components, we can write, for strain increments and total strains, respectively,

$$\dot{\epsilon}_r = \dot{\epsilon}_r^{(e)} + \dot{\epsilon}_r^{(p)}, \quad \dot{\epsilon}_\theta = \dot{\epsilon}_\theta^{(e)} + \dot{\epsilon}_\theta^{(p)}, \quad \dot{\epsilon}_z = \dot{\epsilon}_z^{(e)} + \dot{\epsilon}_z^{(p)} \quad (18)$$

$$\epsilon_r = \epsilon_r^{(e)} + \epsilon_r^{(p)}, \quad \epsilon_\theta = \epsilon_\theta^{(e)} + \epsilon_\theta^{(p)}, \quad \epsilon_z = \epsilon_z^{(e)} + \epsilon_z^{(p)} \quad (19)$$

Now, plane strain requires that the total out-of-plane strain be zero, so from Equations 17 we can conclude that the elastic component is also zero, i.e.

$$\epsilon_z = 0, \quad \epsilon_z^{(e)} = 0, \quad \epsilon_z^{(p)} = 0 \quad (20)$$

Elastic strains are related to stresses by Hooke's law.

$$E\epsilon_r^{(e)} = \sigma_r - \nu(\sigma_\theta + \sigma_z) \quad (21)$$

$$E\epsilon_\theta^{(e)} = \sigma_\theta - \nu(\sigma_z + \sigma_r) \quad (22)$$

$$E\epsilon_z^{(e)} = \sigma_z - \nu(\sigma_r + \sigma_\theta) \quad (23)$$

And, since ϵ_z is zero in the elastic region (outside the elastic-plastic boundary at $r = R$), from Equation 23

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) \quad (24)$$

From the yield condition of Equation 7, circumferential stress at the elastic-plastic boundary can be expressed as a function of the radial stress. Using this and Equation 24, Equations 21 and 22 can be used to express the radial and circumferential elastic strains as functions of radial stress:

$$2G\epsilon_r^{(e)} = [1 - (N+1)\nu]\sigma_r + \nu\sigma_u \quad (25)$$

$$2G\epsilon_{\theta}^{(e)} = [N - (N+1)v]\sigma_r - (1-v)\sigma_u \quad (26)$$

Note that for computational convenience, the shear modulus is substituted for Young's modulus at this point, as allowed by the relationship $E=2G(1+v)$, and strain terms in subsequent equations are frequently expressed in the form of the left-hand side of Equation 26.

For the axisymmetric configuration, the following strain-displacement relationships apply

$$\epsilon_r = \frac{du}{dr}, \quad \epsilon_{\theta} = \frac{u}{r} \quad (27)$$

from which the compatibility equation can be written as

$$r \frac{d\epsilon_{\theta}}{dr} + \epsilon_{\theta} - \epsilon_r = 0 \quad (28)$$

Using Equations 19 to separate strains into elastic and plastic components, Equations 17 to express radial strain in terms of circumferential strain, and Equations 25 and 26 to substitute for the elastic strains, we can rewrite Equation 28 as

$$r \frac{d\epsilon_{\theta}^{(p)}}{dr} + (M+1)\epsilon_{\theta}^{(p)} = \frac{1-v}{2G}(N+1)[(N-1)p_a + \sigma_u] \left(\frac{r}{a}\right)^{N-1} \quad (29)$$

By inspection, it can be seen that this differs from the corresponding expression in Reference 3 only in the substitution of $(M+1)$ for $(N+1)$ on the left-hand side, and indeed degenerates to the identical equation for the case where $M = N$. The solution to Equation 29 is readily determined by direct integration over the interval from $r = r$ to $r = R$. The resultant solution is

$$\epsilon_{\theta}^{(p)} = -\frac{1-v}{2G} \cdot \frac{N+1}{M+N} [(N-1)p_a + \sigma_u] \left(\frac{R}{a}\right)^{N-1} \left[\left(\frac{R}{r}\right)^{M+1} - \left(\frac{r}{R}\right)^{N-1} \right] \quad (30)$$

The counterpart of equation 30 in Reference 3 contains a typographical error in that the second term on the right-hand side should read $\frac{N+1}{2N}$. The plastic component of radial strain can be obtained from Equations 17 although at this

point, R remains unknown.

If the material inside of radius $r = R$ is assumed to be replaced with a pressure p , it follows from Equations 5 that

$$\sigma_r(R) = -p, \quad \sigma_\theta(R) = -2p_b + p, \quad \sigma_z(R) = -2vp_b \quad (31)$$

Since the yield condition of Equation 7 is applicable at $r = R$,

$$p_b = \frac{1}{2}[(N+1)p + \sigma_u] \quad (32)$$

$$\sigma_r(R) = -\frac{2p_b - \sigma_u}{N+1} \quad (33)$$

Equating the radial stress from Equation 33 to the radial stress computed from Equation 14 for $r = R$ yields

$$\left(\frac{R}{a}\right)^{N-1} = \frac{2}{N+1} \cdot \frac{(N-1)p_b + \sigma_u}{(N-1)p_a + \sigma_u} \quad (34)$$

When $(N+1)v > 1$, σ_z will be the intermediate principal stress throughout the plastic zone and the foregoing procedure is adequate to completely define the solution. This corresponds to the Hendron and Aiyer solution as modified by Detournay and St. John to include arbitrary dilatancy.

If $(N+1)v < 1$, the elastic-plastic boundary still forms at $r = R$, but this condition exists only as long as σ_z is the intermediate principal stress, i.e. as long as $\sigma_r < -v\sigma_u/[1-(N+1)v]$. From Equation 14, we see that σ_r is maximum (least compressive) at $r = a$ and minimum (most compressive) at $r = R$. This implies that as the loading increases, a plastic zone forms, in which the radial and out-of-plane stresses are equal. In Figure 1, this is the annular region between $r = \tilde{R}$ and $r = \bar{R}$, where \tilde{R} and \bar{R} are defined as the minimum and maximum, respectively, radii where $\sigma_r = \sigma_z$. At $r = \tilde{R}$ then, we have

$$\sigma_r(\tilde{R}) = \sigma_z(\tilde{R}) = -\frac{v\sigma_u}{1-(N+1)v} \quad (35)$$

and from plane strain considerations, the circumferential stress is

$$\sigma_\theta(\tilde{R}) = -\frac{(1-v)\sigma_u}{1-(N+1)v} \quad (36)$$

We compute the external pressure at which $R = \tilde{R}$ from Equations 31 and 35 as

$$\tilde{p}_b = \frac{\sigma_u}{2[1 - (N+1)v]} \quad (37)$$

which can then be substituted in Equation 34 to compute \tilde{R}

$$\left(\frac{\tilde{R}}{a}\right)^{N-1} = \frac{(1-2v)\sigma_u}{[1 - (N+1)v][(N-1)p_a + \sigma_u]} \quad (38)$$

Note that Equation 38 does not involve \tilde{p}_b , from which we can conclude that as p_b is increased beyond \tilde{p}_b , \tilde{R} remains constant.

3.2 INCREASED LOADING--OUTER PLASTIC ZONE.

As discussed earlier, $\sigma_\theta < \sigma_r < \sigma_z$ at $r = R$. Therefore there must be a third plastic zone between $r = \tilde{R}$ and $r = R$, as shown in Figure 1, which develops when p_b is increased beyond \tilde{p}_b . In this region, the yield condition of Equation 1 applies and the flow rule (with arbitrary dilatancy) is

$$\dot{\epsilon}_r^{(p)} = 0, \quad \dot{\epsilon}_\theta^{(p)} = \lambda, \quad \dot{\epsilon}_z^{(p)} = -M\lambda \quad (39)$$

In the same manner that strain relations were developed from the flow rule of Equations 15, strain relations can be derived from Equations 39 as

$$\epsilon_z^{(p)} + M\epsilon_\theta^{(p)} = 0, \quad \epsilon_r^{(p)} = 0 \quad (40)$$

and from the plane-strain condition,

$$\epsilon_r = \epsilon_r^{(e)}, \quad \epsilon_\theta = \epsilon_\theta^{(e)} + \frac{1}{M}\epsilon_z^{(e)}, \quad \epsilon_z = 0 \quad (41)$$

Also, in the manner that Equations 25 and 26 were developed using the Hooke's law relationships (Equations 21 through 23), Equations 41 can be used with Equation 1 and Hooke's law to obtain expressions for radial and circumferential stress in the outer plastic zone

$$[MN+1 - (M+1)(N+1)v]\sigma_r = [MN+1 - (M+N)v]2G\epsilon_r + M(N+1)v2G\epsilon_\theta + (M-1)v\sigma_u \quad (42)$$

$$[MN+1-(M+1)(N+1)v]\sigma_\theta = N(M+1)v2G\epsilon_r + MN2G\epsilon_\theta - [1-(M+1)v]\sigma_u \quad (43)$$

and with Equations 13 (equilibrium) and 27 (strain-displacement), this results in the following differential equation for radial displacement

$$r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} \left[1 + \frac{(M-N)v}{MN} \beta^2 \right] - \beta^2 u = -\frac{\beta^2}{MN} \cdot \frac{(1-2v)\sigma_u}{2G} r \quad (44)$$

where

$$\beta^2 = \frac{MN}{MN+1-(M+N)v} \quad (45)$$

Note that Equation 44 of Reference 3 contains two typographical errors; the right-hand side should be $\frac{\alpha^2}{N^2} \frac{(1-2v)\sigma_u}{2G} r$. Solving Equation 44 for radial displacement gives

$$2Gu = A_1 r^{\gamma_1} + A_2 r^{-\gamma_2} + A \cdot r \quad (46)$$

The corresponding strains are

$$2G \frac{du}{dr} = 2G\epsilon_r = \gamma_1 A_1 r^{\gamma_1-1} - \gamma_2 A_2 r^{-\gamma_2-1} + A \quad (47)$$

$$2G \frac{u}{r} = 2G\epsilon_\theta = A_1 r^{\gamma_1-1} + A_2 r^{-\gamma_2-1} + A \quad (48)$$

where

$$\gamma_1 = \beta^2 \frac{(N-M)v}{2MN} + \beta^2 \sqrt{\frac{(N-M)^2 v^2}{(MN)^2} + \frac{1}{\beta^2}}, \quad \gamma_2 = \beta^2 \frac{(M-N)v}{2MN} + \beta^2 \sqrt{\frac{(M-N)^2 v^2}{(MN)^2} + \frac{1}{\beta^2}},$$

$$A = -\frac{(1-2v)}{1-2Nv} \sigma_u \quad (49)$$

and A_1 and A_2 are constants, yet to be determined.

Compared to the associated-flow-rule formulations of Reference 3, Equations 46, 47, and 48 involve additional factors and non-identical powers of r . In spite of these complexities, for the case where $M = N$, $\gamma_1 = \gamma_2 = \beta$,

and β is identical to the term α used by Florence and Schwer. Consequently, Equations 46, 47, and 48 become identical to their Reference 3 counterparts.

We can employ the equilibrium and compatibility conditions at the elastic-plastic boundary ($r = R$) to equate radial stresses and circumferential strains across the boundary. From Equations 1 and 5, the elastic-zone radial stress is

$$\sigma_r(R) = -2(1 - Nv)p_b + \sigma_u \quad (50)$$

from which the circumferential strain can be obtained, using Equation 26, as

$$2G\epsilon_\theta(R) = -2(N-1)v p_b - \sigma_u \quad (51)$$

Invoking compatibility of radial displacement (or circumferential strain), we can equate the right-hand sides of Equations 48 and 51, resulting in

$$A_1 R^{\gamma_1-1} + A_2 R^{-\gamma_2-1} = c_1, \quad c_1 = \frac{2(N-1)v}{1-2Nv} [\sigma_u - (1-2Nv)p_b] \quad (52)$$

From equilibrium, the elastic radial stress from Equation 50 is equated to the plastic zone stress from Equation 42, and by substituting strains from Equations 47 and 48

$$b_1 A_1 R^{\gamma_1-1} + b_2 A_2 R^{-\gamma_2-1} = c_2 \quad (53)$$

where

$$b_1 = M(N+1)v + [MN+1 - (M+N)v]\gamma_1,$$

$$b_2 = M(N+1)v - [MN+1 - (M+N)v]\gamma_2,$$

$$c_2 = 2 \frac{1-Nv}{1-2Nv} [MN+1 - (M+1)(N+1)v] [\sigma_u - (1-2Nv)p_b]$$

Since Equations 52 and 53 are linear in A_1 and A_2 , we can solve simultaneously (in terms of unknown R) as

$$A_1 = \frac{c_2 - b_2 c_1}{b_1 - b_2} \left(\frac{1}{R^{\gamma_1-1}} \right) = \frac{B_1}{R^{\gamma_1-1}}, \quad A_2 = \frac{b_1 c_1 - c_2}{b_1 - b_2} \left(\frac{1}{R^{-\gamma_2-1}} \right) = \frac{B_2}{R^{-\gamma_2-1}} \quad (53.1)$$

At the assumed plastic-plastic boundary at $r = \bar{R}$ and in the plastic zone between $r = \tilde{R}$ and $r = \bar{R}$, the radial and out-of-plane stresses are equal and the yield conditions of Equations 1 and 7 are both valid. Using Equation 7, we can now write

$$N\sigma_r(\bar{R}) - \sigma_\theta(\bar{R}) = \sigma_u \quad (54)$$

Since the stresses from Equations 42 and 43 are valid at $r = \bar{R}$, we can substitute them in Equation 54 to get

$$b_3 A_1 \bar{R}^{\gamma_1 - 1} + b_4 A_2 \bar{R}^{-\gamma_2 - 1} = c_3 \frac{1 - 2\nu}{1 - 2N\nu} \sigma_u \quad (55)$$

where

$$b_3 = M[(N+1)\nu - 1] + [MN + 1 - (2M + N + 1)\nu]\gamma_1$$

$$b_4 = M[(N+1)\nu - 1] - [MN + 1 - (2M + N + 1)\nu]\gamma_2$$

$$c_3 = [MN + 1 - (M+1)(N+1)\nu]$$

The expressions for A_1 and A_2 in Equation 53.1 can be substituted in Equation 55 to define the ratio of \bar{R} to R .

$$b_3 B_1 \left(\frac{\bar{R}}{R} \right)^{\gamma_1 - 1} + b_4 B_2 \left(\frac{\bar{R}}{R} \right)^{-\gamma_2 - 1} = c_3 \frac{1 - 2\nu}{1 - 2N\nu} \sigma_u \quad (56)$$

Since the ratio cannot be solved for explicitly, an implicit scheme will be required, but rapid convergence is expected. The radius \bar{R} remains unknown pending consideration of the region from $r = \tilde{R}$ to $r = \bar{R}$.

3.3 INCREASED LOADING--MIDDLE PLASTIC ZONE.

As noted above, both of the yield conditions (Equations 1 and 7) are valid in the region $\tilde{R} < r < \bar{R}$, and are restated here to continue the parallelism with Reference 3.

$$f_1 = \sigma_\theta - N\sigma_r + \sigma_u = 0 \quad (57)$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \quad (58)$$

From Equation 57 we obtain an expression for circumferential stress to substitute into the equilibrium equation (Equation 13). The resulting differential equation is

$$r \frac{d\sigma_r}{dr} + \sigma_r(1-N) = -\sigma_u \quad (58.1)$$

for which the solution is

$$\sigma_r = \sigma_r(\bar{R}) \cdot \left(\frac{r}{\bar{R}}\right)^{N-1} + \left[1 - \left(\frac{r}{\bar{R}}\right)^{N-1}\right] \frac{\sigma_u}{N-1} \quad (59)$$

If $\sigma_r(\bar{R})$ and \bar{R} (from Equations 35 and 38) are substituted in Equation 59, it becomes

$$\sigma_r = - \left[p_a + \frac{\sigma_u}{N-1} \right] \left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \quad (60)$$

Since Equations 57 through 60 are independent of the flow rule, they are identical to their counterparts in Reference 3.

We recall that Equation 42 (radial stress) is valid in the region $\bar{R} < r < R$ and Equation 60 is valid for $\bar{R} < r < \bar{R}$. Since we must assure continuity of radial stress at $r = \bar{R}$, we can equate the two expressions (evaluated at $r = \bar{R}$) to obtain an expression for \bar{R}

$$b_1 A_1 \bar{R}^{\gamma_1-1} + b_2 A_2 \bar{R}^{-\gamma_2-1} = c_3 \frac{N(1-2\nu)}{(N-1)(1-2N\nu)} \sigma_u - c_3 \left(p_a + \frac{\sigma_u}{N-1} \right) \left(\frac{\bar{R}}{a}\right)^{N-1} \quad (61)$$

with b_1 , b_2 , and c_3 as defined in Equations 53 and 55. The expressions for A_1 and A_2 from Equations 53.1 can be substituted in Equation 61:

$$b_1 B_1 \left(\frac{\bar{R}}{R}\right)^{\gamma_1-1} + b_2 B_2 \left(\frac{\bar{R}}{R}\right)^{-\gamma_2-1} = c_3 \frac{N(1-2\nu)}{(N-1)(1-2N\nu)} \sigma_u - c_3 \left(p_a + \frac{\sigma_u}{N-1} \right) \left(\frac{\bar{R}}{a}\right)^{N-1} \quad (62)$$

Since \bar{R}/R was determined in Equation 56, \bar{R} can be computed from Equation 62. R can then be obtained from \bar{R}/R . Then, A_1 and A_2 can be calculated

explicitly from Equations 53.1, which is repeated here as Equation 63, again to continue the parallelism with Reference 3.

$$A_1 = \frac{c_2 - b_2 c_1}{b_1 - b_2} \left(\frac{1}{R^{\gamma_1 - 1}} \right) = \frac{B_1}{R^{\gamma_1 - 1}}, \quad A_2 = \frac{b_1 c_1 - c_2}{b_1 - b_2} \left(\frac{1}{R^{\gamma_2 - 1}} \right) = \frac{B_2}{R^{\gamma_2 - 1}} \quad (63)$$

Strains and displacements in the region $\bar{R} < r < R$ can then be computed from Equations 46 through 48; stresses are available from Equations 1, 42, and 43.

For the yield conditions of Equations 57 and 58, the flow rule with arbitrary dilatancy is

$$\dot{\epsilon}_r^{(p)} = \lambda_1 \frac{\partial f_1}{\partial \sigma_r} + \lambda_2 \frac{\partial f_2}{\partial \sigma_r} = -M\lambda_1 \quad (64)$$

$$\dot{\epsilon}_\theta^{(p)} = \lambda_1 \frac{\partial f_1}{\partial \sigma_\theta} + \lambda_2 \frac{\partial f_2}{\partial \sigma_\theta} = \lambda_1 + \lambda_2 \quad (65)$$

$$\dot{\epsilon}_z^{(p)} = \lambda_1 \frac{\partial f_1}{\partial \sigma_z} + \lambda_2 \frac{\partial f_2}{\partial \sigma_z} = -M\lambda_2 \quad (66)$$

from which

$$g(r) = \epsilon_r^{(p)} + M\epsilon_\theta^{(p)} + \epsilon_z^{(p)} = 0 \quad (67)$$

We designate the plastic strains as $\bar{\epsilon}_r^{(p)}$, $\bar{\epsilon}_\theta^{(p)}$, $\bar{\epsilon}_z^{(p)}$ when $r = \bar{R}$, so that

$$g(r) = \bar{\epsilon}_r^{(p)} + M\bar{\epsilon}_\theta^{(p)} + \bar{\epsilon}_z^{(p)} \quad (68)$$

Since Equation 40 is applicable in the region $\bar{R} < r < R$, $g(r) = 0$, from which

$$\epsilon_r^{(p)} + M\epsilon_\theta^{(p)} + \epsilon_z^{(p)} = 0 \quad (69)$$

and with $\epsilon_z^{(e)} = -\epsilon_z^{(p)}$ from the plane strain condition,

$$\epsilon_r = -M\epsilon_\theta + M\epsilon_\theta^{(e)} + \epsilon_r^{(e)} + \epsilon_z^{(e)} \quad (70)$$

With $\sigma_r = \sigma_z$ given by Equation 60 and using the yield condition of Equation 57 we can write Hooke's law from Equations 21 through 23 as

$$E\varepsilon_r^{(e)} = E\varepsilon_z^{(e)} = [1 - (N+1)v]\sigma_r + v\sigma_u \quad (71)$$

$$E\varepsilon_\theta^{(e)} = (N-2v)\sigma_r - \sigma_u \quad (72)$$

Elastic strains from Equations 71 and 72 can be substituted into Equation 70. Substitution of the radial strain expression thus obtained into the compatibility equation (Equation 28) yields a differential equation in terms of the circumferential strain

$$2G \frac{d}{dr} (r^{N+1} \varepsilon_\theta) = - \frac{MN+2-2(M+N+1)v}{(N-1)(1+v)} [(N-1)p_a + \sigma_u] \frac{r^{M+N-1}}{a^{N-1}} + \frac{(M+2)(1-2v)\sigma_u}{(N-1)(1+v)} r^M \quad (73)$$

This differential equation is directly integrable, and integration over the limits of r to \bar{R} results in the following expression for ε_θ

$$2G\varepsilon_\theta = 2G\varepsilon_\theta(\bar{R}) \cdot \left(\frac{\bar{R}}{r}\right)^{N+1} + \frac{MN+2-2(M+N+1)v}{(M+N)(N-1)(1+v)} [(N-1)p_a + \sigma_u] \left[\left(\frac{\bar{R}}{r}\right)^{M+N} - 1 \right] \left(\frac{r}{a}\right)^{N-1} - \frac{(M+2)(1-2v)\sigma_u}{(N-1)(M+1)(1+v)} \left[\left(\frac{\bar{R}}{r}\right)^{N+1} - 1 \right] \quad (74)$$

$\varepsilon_\theta(\bar{R})$ can be obtained from Equation 48 since we now have values for A_1 and A_2 . The radial strain is obtained from Equations 70 through 72 giving

$$2G\varepsilon_r = -2G\varepsilon_\theta + \frac{MN+2-2(M+N+1)v}{1+v} \sigma_r - \frac{M-2v}{1+v} \sigma_u \quad (75)$$

3.4 INCREASED LOADING--INNER PLASTIC ZONE.

For an applied pressure in excess of \bar{p}_b , stresses in the inner plastic zone, $a < r < \bar{R}$, remain constant. Radial stress is given by Equation 14 and the radial and out-of-plane plastic strain components are given by Equations 17. Equation 29 is valid in this region, but we must perform the integration

from r to \tilde{R} for this condition, which yields the circumferential plastic strain as

$$2G\epsilon_{\theta}^{(p)} = 2G\epsilon_{\theta}^{(p)}(\tilde{R})\left(\frac{\tilde{R}}{r}\right)^{M+1} - \frac{(N+1)(1-\nu)}{M+N}[(N-1)p_a + \sigma_u]\left(\frac{\tilde{R}}{a}\right)^{N-1}\left[\left(\frac{\tilde{R}}{r}\right)^{M+1} - \left(\frac{r}{\tilde{R}}\right)^{N-1}\right] \quad (76)$$

Elastic strains are available from Equations 20 through 26. In order to compute the circumferential strain at $r = \tilde{R}$ from Equation 76, we can compute the total strain from Equation 74 evaluated at $r = \tilde{R}$ and subtract the elastic component. Circumferential and out-of-plane stresses are determined using Equations 7 and 24.

SECTION 4

CASE II SOLUTION

As stated earlier, Florence and Schwer presented the solution for Case II in Reference 4. Since that reference is a "stand-alone" paper, they repeated some, but not all, of the equations from Reference 3 with different equation numbers. Consequently, from this point forward, we do not use the Florence and Schwer equation numbers, but we continue to follow their format. For the convenience of the reader, several of the equations developed earlier are repeated in this section.

As discussed in both References 3 and 4, there are at most two plastic zones for Case II. The problem geometry is shown in Figure 2 (from Reference 4).

4.1 INITIAL GROWTH OF THE PLASTIC ZONE.

As shown in Table 2, the order of the principal stresses at initial yield is $\sigma_\theta < \sigma_r < \sigma_z$ for Case II. Therefore, the yield condition of Equation 1 applies.

$$f = \sigma_\theta - N\sigma_z + \sigma_u = 0 \quad (1)$$

For this condition, with arbitrary dilatancy, the flow rule was given in Equations 39,

$$\dot{\epsilon}_r^{(p)} = 0, \quad \dot{\epsilon}_\theta^{(p)} = \lambda, \quad \dot{\epsilon}_z^{(p)} = -M\lambda \quad (39)$$

from which the plastic strain relationships, given by Equations 40,

$$\epsilon_z^{(p)} + M\epsilon_\theta^{(p)} = 0, \quad \epsilon_r^{(p)} = 0 \quad (40)$$

were derived. Incremental and total strains are given by Equations 18 and 19,

$$\dot{\epsilon}_r = \dot{\epsilon}_r^{(e)} + \dot{\epsilon}_r^{(p)}, \quad \dot{\epsilon}_\theta = \dot{\epsilon}_\theta^{(e)} + \dot{\epsilon}_\theta^{(p)}, \quad \dot{\epsilon}_z = \dot{\epsilon}_z^{(e)} + \dot{\epsilon}_z^{(p)} \quad (18)$$

$$\epsilon_r = \epsilon_r^{(e)} + \epsilon_r^{(p)}, \quad \epsilon_\theta = \epsilon_\theta^{(e)} + \epsilon_\theta^{(p)}, \quad \epsilon_z = \epsilon_z^{(e)} + \epsilon_z^{(p)} \quad (19)$$

from which the three components of total strain, given by Equations 41,

$$\epsilon_r = \epsilon_r^{(e)}, \quad \epsilon_\theta = \epsilon_\theta^{(e)} + \frac{1}{M} \epsilon_z^{(e)}, \quad \epsilon_z = 0 \quad (41)$$

are obtained. The yield condition of Equation 1 allowed us to eliminate σ_z from the Hooke's law expressions of Equations 21, 22, and 23,

$$E\epsilon_r^{(e)} = \sigma_r - \nu(\sigma_\theta + \sigma_z) \quad (21)$$

$$E\epsilon_\theta^{(e)} = \sigma_\theta - \nu(\sigma_z + \sigma_r) \quad (22)$$

$$E\epsilon_z^{(e)} = \sigma_z - \nu(\sigma_r + \sigma_\theta) \quad (23)$$

which led to the expressions for radial and circumferential stress in terms of strain in Equations 42 and 43.

$$[MN+1-(M+1)(N+1)\nu]\sigma_r = [MN+1-(M+N)\nu]2G\epsilon_r + M(N+1)\nu 2G\epsilon_\theta + (M-1)\nu\sigma_u \quad (42)$$

$$[MN+1-(M+1)(N+1)\nu]\sigma_\theta = N(M+1)\nu 2G\epsilon_r + MN 2G\epsilon_\theta - [1-(M+1)\nu]\sigma_u \quad (43)$$

Equation 13,

$$r \frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = 0 \quad (13)$$

(equilibrium) and Equations 27,

$$\epsilon_r = \frac{du}{dr}, \quad \epsilon_\theta = \frac{u}{r} \quad (27)$$

(strain-displacement) were then used to obtain the differential equation of displacement given by

$$r^2 \frac{d^2u}{dr^2} + r \frac{du}{dr} \left[1 + \frac{(M-N)\nu}{MN} \beta^2 \right] - \beta^2 u = -\frac{\beta^2}{MN} \cdot \frac{(1-2\nu)\sigma_u}{2G} r \quad (44)$$

The solution of Equation 44 is

$$2Gu = A_1 r^{\gamma_1} + A_2 r^{-\gamma_2} + A \cdot r \quad (46)$$

and the associated strain relationships are

$$2G \frac{du}{dr} = 2G\epsilon_r = \gamma_1 A_1 r^{\gamma_1-1} - \gamma_2 A_2 r^{-\gamma_2-1} + A \quad (47)$$

$$2G \frac{u}{r} = 2G\epsilon_\theta = A_1 r^{\gamma_1-1} + A_2 r^{-\gamma_2-1} + A \quad (48)$$

Continuity of displacement (or circumferential strain) and radial stress at the elastic-plastic boundary ($r = R$) then allowed us to write Equations 52 and 53.

$$A_1 R^{\gamma_1-1} + A_2 R^{-\gamma_2-1} = c_1, \quad c_1 = \frac{2(N-1)v}{1-2Nv} [\sigma_u - (1-2Nv)p_b] \quad (52)$$

$$b_1 A_1 R^{\gamma_1-1} + b_2 A_2 R^{-\gamma_2-1} = c_2 \quad (53)$$

where

$$b_1 = M(N+1)v + [MN+1 - (M+N)v]\gamma_1,$$

$$b_2 = M(N+1)v - [MN+1 - (M+N)v]\gamma_2,$$

$$c_2 = 2 \frac{1-Nv}{1-2Nv} [MN+1 - (M+1)(N+1)v] [\sigma_u - (1-2Nv)p_b]$$

Expressions for stress and strain in the elastic zone were developed

$$\sigma_r(R) = -2(1-Nv)p_b + \sigma_u \quad (50)$$

$$2G\epsilon_\theta(R) = -2(N-1)v p_b - \sigma_u \quad (51)$$

In Equation 53.1,

$$A_1 = \frac{c_2 - b_2 c_1}{b_1 - b_2} \left(\frac{1}{R^{\gamma_1-1}} \right) = \frac{B_1}{R^{\gamma_1-1}}, \quad A_2 = \frac{b_1 c_1 - c_2}{b_1 - b_2} \left(\frac{1}{R^{-\gamma_2-1}} \right) = \frac{B_2}{R^{-\gamma_2-1}} \quad (53.1)$$

expressions were found for the unknown constants A_1 and A_2 in terms of the still undetermined radius R to the elastic-plastic boundary.

Note that to this point we have merely reiterated conditions developed in Section 3. For Case II, substitution of the negative of the internal pressure $-p_a$ for σ_r and the radial and circumferential strains from Equations 47 and 48 into Equation 42 yields

$$b_1 A_1 a^{\gamma_1-1} + b_2 A_2 a^{\gamma_2-1} = c_3 \left[\frac{\sigma_u}{1-2N\nu} - p_a \right] \quad (77)$$

where the terms b_1 , b_2 , and c_3 are as defined in Equations 53 and 55.

Using the definitions of A_1 and A_2 from Equation 53.1 gives the following equation from which the radius R to the elastic-plastic boundary can be calculated.

$$b_1 B_1 \left(\frac{a}{R} \right)^{\gamma_1-1} + b_2 B_2 \left(\frac{a}{R} \right)^{\gamma_2-1} = c_3 \left[\frac{\sigma_u}{1-2N\nu} - p_a \right] \quad (78)$$

Note that the corresponding equation in Reference 4 (Equation 38) appears to contain a typographical error; the second term exponent should be $1+\alpha$.

As long as $\sigma_r < \sigma_z$ (the outer plastic zone of Figure 2), radial and circumferential strains can be computed from Equations 47 and 48. Equations 42 and 43 can then be used to get the corresponding stresses.

4.2 INCREASED LOADING.

As with Case I, it is possible to increase the external load to a magnitude such that the out-of-plane stress will equal the radial stress. This condition results in formation of the inner plastic zone of Figure 2. In this situation, the radius R can no longer be obtained from Equation 78.

As p_b is increased, the inner plastic zone begins to form when the out-of-plane stress first equals the radial stress at the edge of the opening, i. e., at $r = a$. The far-field pressure at this instant is defined as p'_b , and the corresponding radius to the elastic-plastic boundary is defined as R' . The development of expressions for these terms is described in the following paragraphs.

The expressions for radial and circumferential stress in the outer plastic zone were presented in Equations 42 and 43. Because stress continuity is required across the boundary between the two plastic zones,

these equations hold at $r = a$ when $P_b = P'_b$. Upon invoking the definition of c_3 from Equation 55, Equations 42 and 43 become

$$c_3 \sigma_r = [MN + 1 - (M + N)v]2G\epsilon_r + M(N + 1)v2G\epsilon_\theta + (M - 1)v\sigma_u \quad (79)$$

$$c_3 \sigma_\theta = N(M + 1)v2G\epsilon_r + MN2G\epsilon_\theta - [1 - (M + 1)v]\sigma_u \quad (80)$$

The out-of-plane stress from the yield condition of Equation 1 is

$$\sigma_z = \frac{\sigma_\theta}{N} + \frac{\sigma_u}{N} \quad (81)$$

Combining Equations 80 and 81 yields

$$c_3 \sigma_z = (M + 1)v2G\epsilon_r + M2G\epsilon_\theta - \frac{1}{N}[1 - (M + 1)v - c_3]\sigma_u \quad (82)$$

At the edge of the opening, i. e., at $r = a$, $\sigma_r = \sigma_z = -p_a$, and strains $\epsilon_r(a)$ and $\epsilon_\theta(a)$ may be obtained from Equations 47 and 48. When these substitutions are made, Equations 79 and 82 become

$$\begin{aligned} -c_3 p_a = [MN + 1 - (M + N)v] & \left[\gamma_1 c'_1 B_3 \left(\frac{a}{R'} \right)^{\gamma_1 - 1} - \gamma_2 c'_1 B_4 \left(\frac{a}{R'} \right)^{-\gamma_2 - 1} + A \right] \\ & + [M(N + 1)v] \left[c'_1 B_3 \left(\frac{a}{R'} \right)^{\gamma_1 - 1} + c'_1 B_4 \left(\frac{a}{R'} \right)^{-\gamma_2 - 1} + A \right] + (M - 1)v\sigma_u \end{aligned} \quad (83)$$

$$\begin{aligned} -c_3 p_a = [(M + 1)v] & \left[\gamma_1 c'_1 B_3 \left(\frac{a}{R'} \right)^{\gamma_1 - 1} - \gamma_2 c'_1 B_4 \left(\frac{a}{R'} \right)^{-\gamma_2 - 1} + A \right] \\ & + M \left[c'_1 B_3 \left(\frac{a}{R'} \right)^{\gamma_1 - 1} + c'_1 B_4 \left(\frac{a}{R'} \right)^{-\gamma_2 - 1} + A \right] - \frac{1}{N}[1 - (M + 1)v - c_3]\sigma_u \end{aligned} \quad (84)$$

where the constants B_3 and B_4 are obtained by rearranging the expressions for B_1 and B_2 (Equations 53.1)

$$c_1 B_3 = \frac{c_1}{b_1 - b_2} \left[\frac{c_3(1 - Nv)}{(N - 1)v} - b_2 \right] \quad (85)$$

$$c_1 B_4 = \frac{c_1}{b_1 - b_2} \left[b_1 - \frac{c_3(1 - Nv)}{(N - 1)v} \right] \quad (86)$$

The term c'_1 is defined as the value of c_1 (from Equation 52) when $p_b = p'_b$

$$c'_1 = \frac{2(N-1)v}{1-2Nv} [\sigma_u - (1-2Nv)p'_b] \quad (87)$$

Equations 83 and 84 may be solved simultaneously and, after invoking the definition of c'_1 from Equation 87, we obtain

$$\frac{a}{R'} = \left[\frac{B_3 \cdot \{2v\sigma_u - (1-2Nv)p_a\}b_1 - \{\sigma_u - (1-2Nv)p_a\}\{M + (M+1)v\gamma_1\}}{B_4 \cdot \{2v\sigma_u - (1-2Nv)p_a\}b_2 - \{\sigma_u - (1-2Nv)p_a\}\{M - (M+1)v\gamma_2\}} \right]^{\frac{1}{-\gamma_1 - \gamma_2}} \quad (88)$$

and

$$p_b = \frac{\sigma_u}{1-2Nv} - \frac{c_3 \left(\frac{\sigma_u}{1-2Nv} - p_a \right)}{2(N-1)v \left[b_1 B_3 \left(\frac{a}{R'} \right)^{\gamma_1-1} + b_2 B_4 \left(\frac{a}{R'} \right)^{-\gamma_2-1} \right]} \quad (89)$$

Since p'_b is the minimum far-field pressure that will cause a second plastic zone to form, this equation can be used to determine whether one or two plastic zones will form.

The maximum radius at which the radial and out-of-plane stresses are equal was defined earlier as \bar{R} . An expression for the ratio of \bar{R} to R can be developed in a fashion similar to that employed in the development of Equation 56 for Case I.

The stresses at $r = \bar{R}$ can be expressed by

$$\sigma_z(\bar{R}) = \sigma_r(\bar{R}) \quad (90)$$

As stated above, the out-of-plane stress from the yield condition of Equation 1 is

$$\sigma_z = \frac{\sigma_\theta}{N} + \frac{\sigma_u}{N} \quad (81)$$

Circumferential stress (at $r = \bar{R}$) from Equation 43 is substituted in Equation

81, which is then equated to the radial stress from Equation 42 to get

$$[M(N-2\nu) - N\nu + (1-\nu)]2G\epsilon_r + M[N\nu - (1-\nu)]2G\epsilon_\theta = M(1-2\nu)\sigma_u \quad (91)$$

Substitution of strains from Equations 47 and 48 in Equation 91 gives

$$b_3 A_1 \bar{R}^{\gamma_1-1} + b_4 A_2 \bar{R}^{-\gamma_2-1} = c_3 \frac{1-2\nu}{1-2N\nu} \sigma_u \quad (92)$$

Equation 92 is identical to Equation 55 and the terms b_3 , b_4 and c_3 were defined in conjunction with that equation. Using the definitions of A_1 and A_2 from Equations 53.1 yields an expression for the desired ratio

$$b_3 B_1 \left(\frac{\bar{R}}{R} \right)^{\gamma_1-1} + b_4 B_2 \left(\frac{\bar{R}}{R} \right)^{-\gamma_2-1} = c_3 \frac{1-2\nu}{1-2N\nu} \sigma_u \quad (93)$$

which is identical to Equation 56. As with Case I, \bar{R} remains unknown pending consideration of the region where $\sigma_r = \sigma_z$. Also, as with Case I, both yield conditions (Equations 57 and 58) are applicable when $\sigma_r = \sigma_z$ (the inner plastic zone in this case).

$$f_1 = \sigma_\theta - N\sigma_r + \sigma_u = 0 \quad (57)$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \quad (58)$$

Using Equation 57 and the equilibrium expression of Equation 13

$$r \frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = 0 \quad (13)$$

Radial stress can be expressed by Equation 60.

$$\sigma_r = - \left[p_a + \frac{\sigma_u}{N-1} \right] \left(\frac{r}{a} \right)^{N-1} + \frac{\sigma_u}{N-1} \quad (60)$$

In order to satisfy radial stress continuity at $r = \bar{R}$, we again set Equation 60 equal to Equation 42 and substitute strains from Equations 47 and 48, all quantities being evaluated at $r = \bar{R}$. As with Case I, this yields Equation 62.

$$b_1 B_1 \left(\frac{\bar{R}}{R} \right)^{\gamma_1-1} + b_2 B_2 \left(\frac{\bar{R}}{R} \right)^{-\gamma_2-1} = c_3 \frac{N(1-2\nu)}{(N-1)(1-2N\nu)} \sigma_u - c_3 \left(p_a + \frac{\sigma_u}{N-1} \right) \left(\frac{\bar{R}}{a} \right)^{N-1} \quad (62)$$

Since \bar{R}/R was determined by Equation 93, \bar{R} can be computed from Equation 62, as was done in Case I, and R can then be obtained explicitly.

The flow rule expressed by Equations 64, 65, and 66

$$\dot{\epsilon}_r^{(p)} = \lambda_1 \frac{\partial f_1}{\partial \sigma_r} + \lambda_2 \frac{\partial f_2}{\partial \sigma_r} = -M\lambda_1 \quad (64)$$

$$\dot{\epsilon}_\theta^{(p)} = \lambda_1 \frac{\partial f_1}{\partial \sigma_\theta} + \lambda_2 \frac{\partial f_2}{\partial \sigma_\theta} = \lambda_1 + \lambda_2 \quad (65)$$

$$\dot{\epsilon}_z^{(p)} = \lambda_1 \frac{\partial f_1}{\partial \sigma_z} + \lambda_2 \frac{\partial f_2}{\partial \sigma_z} = -M\lambda_2 \quad (66)$$

again leads to the plastic strain expression of Equation 67:

$$g(r) = \epsilon_r^{(p)} + M\epsilon_\theta^{(p)} + \epsilon_z^{(p)} = 0 \quad (67)$$

At $r = \bar{R}$, the plastic strain expression of Equation 68 follows

$$g(r) = \bar{\epsilon}_r^{(p)} + M\bar{\epsilon}_\theta^{(p)} + \bar{\epsilon}_z^{(p)} \quad (68)$$

which, with Equations 40 (which hold in the outer plastic zone),

$$\epsilon_z^{(p)} + M\epsilon_\theta^{(p)} = 0, \quad \epsilon_r^{(p)} = 0 \quad (40)$$

leads to Equation 69:

$$\epsilon_r^{(p)} + M\epsilon_\theta^{(p)} + \epsilon_z^{(p)} = 0 \quad (69)$$

Invoking the plane strain constraint, radial strain can be expressed by Equation 70

$$\epsilon_r = -M\epsilon_\theta + M\epsilon_\theta^{(e)} + \epsilon_r^{(e)} + \epsilon_z^{(e)} \quad (70)$$

Since $\sigma_r = \sigma_z$ in the inner plastic zone, the Case I derivation of Equations 71 and 72 also holds.

$$E\epsilon_r^{(e)} = E\epsilon_z^{(e)} = [1 - (N+1)\nu]\sigma_r + \nu\sigma_\theta \quad (71)$$

$$E\epsilon_{\theta}^{(e)} = (N-2\nu)\sigma_r - \sigma_u \quad (72)$$

As with Case I, a differential equation for the circumferential stress in terms of r is given by Equation 73 when elastic stresses from Equations 71 and 72 and radial strain from Equation 70 are substituted in the compatibility expression of Equation 28.

$$2G \frac{d}{dr} (r^{M+1} \epsilon_{\theta}) = - \frac{MN+2-2(M+N+1)\nu}{(N-1)(1+\nu)} [(N-1)p_a + \sigma_u] \frac{r^{M+N-1}}{a^{N-1}} + \frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(1+\nu)} r^M \quad (73)$$

Again, Equation 73 can be solved by direct integration from r to \bar{R} to get the expression for circumferential strain in Equation 74. As before, $\epsilon_{\theta}(\bar{R})$ is obtained from Equation 48. Radial strain is given by Equation 75 which was obtained from Equations 70, 71, and 72.

$$2G\epsilon_{\theta} = 2G\epsilon_{\theta}(\bar{R}) \cdot \left(\frac{\bar{R}}{r}\right)^{M+1} + \frac{MN+2-2(M+N+1)\nu}{(M+N)(N-1)(1+\nu)} [(N-1)p_a + \sigma_u] \left[\left(\frac{\bar{R}}{r}\right)^{M+N} - 1 \right] \left(\frac{r}{a}\right)^{N-1} - \frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(M+1)(1+\nu)} \left[\left(\frac{\bar{R}}{r}\right)^{M+1} - 1 \right] \quad (74)$$

$$2G\epsilon_r = -2G\epsilon_{\theta} + \frac{MN+2-2(M+N+1)\nu}{1+\nu} \sigma_r - \frac{M-2\nu}{1+\nu} \sigma_u \quad (75)$$

Florence and Schwer (Reference 4) determined that for Case II with an associated flow rule, an inner plastic zone, where σ_z was the intermediate principal stress, did not form. Since our equations for stress in the inner plastic zone are identical to theirs (i.e., they do not involve M), their determination holds for the case of arbitrary dilatancy.

SECTION 5

APPLICATION OF CLOSED-FORM SOLUTIONS

In Section 5.1, procedures for applying the solutions developed in the foregoing sections are discussed. Some numerical examples are presented in Section 5.2.

5.1 PROCEDURES.

The solutions for problems of axisymmetric compression of Mohr-Coulomb materials with arbitrary dilatancy are quite complex, as are the procedures for applying them. Consequently, we have developed a "road map" that is intended to aid the reader, first in determining the appropriate solution case (i. e., Case I or Case II yielding), and then in selecting the proper equations to define the stress and strain fields.

The proper solution case may be determined from the conditional inequalities listed earlier in Table 2. These inequalities involve only the quantities p_a , p_b , v , σ_u , and ϕ (or N). These and the other parameters that must be known (either given or assumed) at the beginning of the problem are listed in Table 3. The process is detailed in Table 4. The decision logic of Table 4 is illustrated schematically in Figure 3. Note that the process of determining the proper yield condition is independent of the dilatancy parameter M .

As may be seen from Table 4 and Figure 3, there are four possible outcomes: (1) the Mohr-Coulomb medium yields everywhere, (2) the medium is elastic everywhere, (3) the medium yields under Case I conditions, or (4) the medium yields under Case II conditions. In both Cases I and II, the far-field medium remains elastic. Depending upon which condition prevails, the reader is directed to Table 5, 6, 7, or 8 for the appropriate step-by-step procedure. Because of the length of these tables, they are located at the end of this section.

As indicated in Table 4 and Figure 3, if under Case I, $(N+1)v \geq 1$, only a single plastic zone will form, regardless of the value of $p_b \geq \bar{p}_b$. This condition follows from the fact that σ_z will be the intermediate principal stress throughout the plastic zone when $(N+1)v \geq 1$. On the other hand, if $(N+1)v < 1$, the three plastic zones shown in Figure 1 will form as p_b is increased.

Table 3. Known parameters.

The following parameters must be either given or assumed:

Unconfined compressive strength σ_u

Friction angle ϕ (or $N = \frac{1+\sin\phi}{1-\sin\phi}$)

Poisson's ratio ν

Arbitrary dilatancy factor M , $1 \leq M \leq N$

Internal pressure p_a

External pressure p_b

Shear modulus G (or $E = 2G(1+\nu)$)

Interior radius a (not required when results are normalized with respect to radius)

Table 4. Process for determining proper step-by-step procedure to follow.

If $Nv < \frac{1}{2}$ and $p_a > \frac{\sigma_u}{1-2Nv}$, then the far-field yields everywhere and the solution is not valid--quit.

Otherwise, compute $(N+1)v$

If $(N+1)v \geq 1$, Case I yielding will occur for $p_b \geq \bar{p}_b$. Use the single-plastic-zone procedure of Table 6.

Otherwise (i.e., $(N+1)v < 1$), compute $\frac{v\sigma_u}{1-(N+1)v}$

If $p_a < \frac{v\sigma_u}{1-(N+1)v}$, Case I yielding will occur for $p_b \geq \bar{p}_b$. Use the three-plastic-zone procedure of Table 7.

Otherwise, Case II yielding will occur for $p_b \geq \bar{p}_b$. Use the procedure of Table 8.

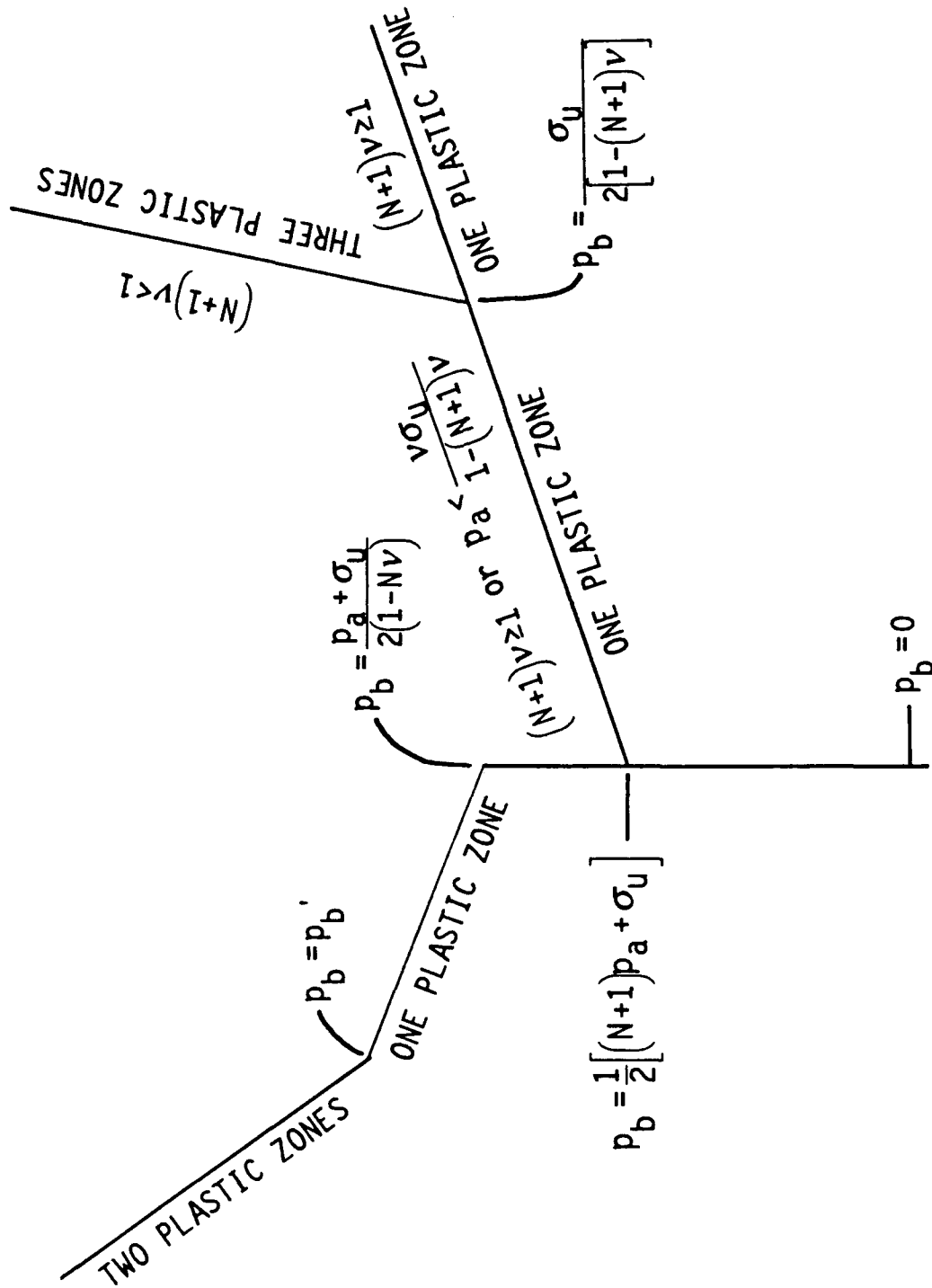


Figure 3. Schematic representation of decision logic.

In the three-plastic-zone case, only a single plastic zone will form when the far-field pressure is small, i. e., when $\bar{p}_b \leq p_b < \tilde{p}_b$. However, there will not be a situation under Case I where two plastic zones form. This is true because the middle and outer plastic zones begin to form simultaneously when $p_b = \tilde{p}_b$ because $R = \bar{R} = \tilde{R}$ at that instant.

Under Case II yielding, the general situation is two plastic zones as shown in Figure 2. However, as with Case I, only a single plastic zone will exist at low far-field pressures, i. e., when $\bar{p}_b \leq p_b < p'_b$.

In the step-by-step procedures (Tables 5 through 8) the equations developed in the foregoing sections are arranged in order to allow one to expeditiously determine the stress and strain fields in the medium surrounding the tunnel. Appropriate amplifying comments have also been included in the tables. In general, each procedure begins with the computation of required constants, followed by the determination of the plastic-plastic and elastic-plastic radii, and concludes with the equations necessary to completely define the stress and strain fields in each zone. For the reader who is interested in determining only the circumferential strain at the tunnel wall (equivalent to the negative of tunnel closure ($\Delta D/D$) in axisymmetric problems of this type), the special case for ϵ_θ at $r = a$ is presented at the end of each table.

Equation numbers from the foregoing sections have been repeated in Tables 5 through 8. A primed equation number indicates that the equation has been modified in some way. Where two equations have been combined, the numbers of both are given. Equations that appear in the tables without numbers are unique to the step-by-step procedures. In the tables, we have continued the use of twice the shear modulus times the strain, rather than expressing the strains explicitly.

5.2 NUMERICAL EXAMPLES.

Two numerical examples are presented, using the input parameters specified by Florence and Schwer in References 3 and 4. In addition to indicating some of the effects of arbitrary dilatancy, these examples illustrate the process for determining the appropriate yield case.

5.2.1 Case I Example.

For this example, input parameters, taken from Florence and Schwer

(Reference 3), are:

Unconfined compressive strength, $\sigma_u = 2,000$ psi (13.8 MPa)

Friction angle, $\phi = 30^\circ$, $N = 3$

Poisson's ratio, $\nu = 0.2$

Internal pressure, $p_a = 500$ psi (3.45 MPa)

Far-field pressure, $p_b = 6,000$ psi (41.4 MPa)

Shear modulus, $G = 10^6$ psi (6,895 MPa)

Interior radius, a (not specified--results presented as functions of r/a)

Three different values of the arbitrary dilatancy parameter are used. These are (1) $M = N$, (2) $M = 1$, and (3) $M = (N + 1)/2$.

The first step is to determine the appropriate step-by-step procedure. Starting at the top of Table 4, $N\nu = 0.6 > \frac{1}{2}$, so the solution procedure is valid.

Next, $(N + 1)\nu = 0.8$. Therefore, $(N + 1)\nu < 1$. In addition,

$$\frac{\nu\sigma_u}{1-(n+1)\nu} = 800$$

and

$$p_a = 500 < 800$$

which indicates that the Case I yield condition applies, so the procedures of Table 7 should be used.

The first step in Table 7 is to determine whether the specified value of p_b will cause yielding to occur and, if so, whether one or three plastic zones will form. Therefore,

$$p_b = 6,000 > \frac{1}{2}[(N+1)p_a + \sigma_u] = 2,000$$

which indicates that yielding occurs. Next

$$\frac{\sigma_u}{2[1-(N+1)\nu]} = 5,000 < p_b = 6,000$$

Therefore, p_b is sufficiently large to cause the formation of three plastic

zones.

For the upper-bound case where $M = N$, calculated radius ratios are (1) $\bar{R}/a = \sqrt{2}$, (2) $\bar{R}/a = 1.527$, and (3) $R/a = 1.598$. Stresses and strains for this case are plotted in Figure 4. Except for differences in scale, these plots are identical to the Florence and Schwer results (Figures 2 and 3 of Reference 3). Contrary to the established sign convention, we also plotted all compressive stresses in Figure 4 (and subsequent figures) as positive to facilitate comparisons with the Florence and Schwer results. Tensile strains are shown as positive, both here and in Reference 3.

For the lower-bound case where $M = 1$, the calculated radius ratios are (1) $\bar{R}/a = \sqrt{2}$, (2) $\bar{R}/a = 1.523$, and (3) $R/a = 1.596$. For the intermediate case where $M = (N + 1)/2 = 2$, the ratios are (1) $\bar{R}/a = \sqrt{2}$, (2) $\bar{R}/a = 1.526$, and (3) $R/a = 1.598$.

The ratio \bar{R}/a remains constant as M is varied since, as may be seen in Equation 38, it is not a function of M . The other two ratios change slightly when M is increased from $M = 1$ to $M = N$.

Stresses and strains for all three cases are plotted in Figure 5. Note that the horizontal scale is more than double that of Figure 4. For this example, there are no discernible differences in the stress plots. Strains differ significantly at the edge of the opening, but are nearly identical, regardless of the value of M , beyond r/a of about 1.5.

5.2.2 Case II Example.

Input parameters for this example were taken from Florence and Schwer (Reference 4):

Unconfined compressive strength, $\sigma_u = 3,000$ psi (20.7 MPa)

Friction angle, $\phi = 19.5^\circ$, $N = 2$

Poisson's ratio, $\nu = 0.2$

Internal pressure, $p_a = 2,000$ psi (13.8 MPa)

Far-field pressure, $p_b = 6,000$ psi (41.4 MPa)

Shear modulus, $G = 0.5 \times 10^6$ psi (3,447 MPa)

Interior radius, a (not specified--results presented as functions of r/a)

Again, three different values of the arbitrary dilatancy parameter are used:

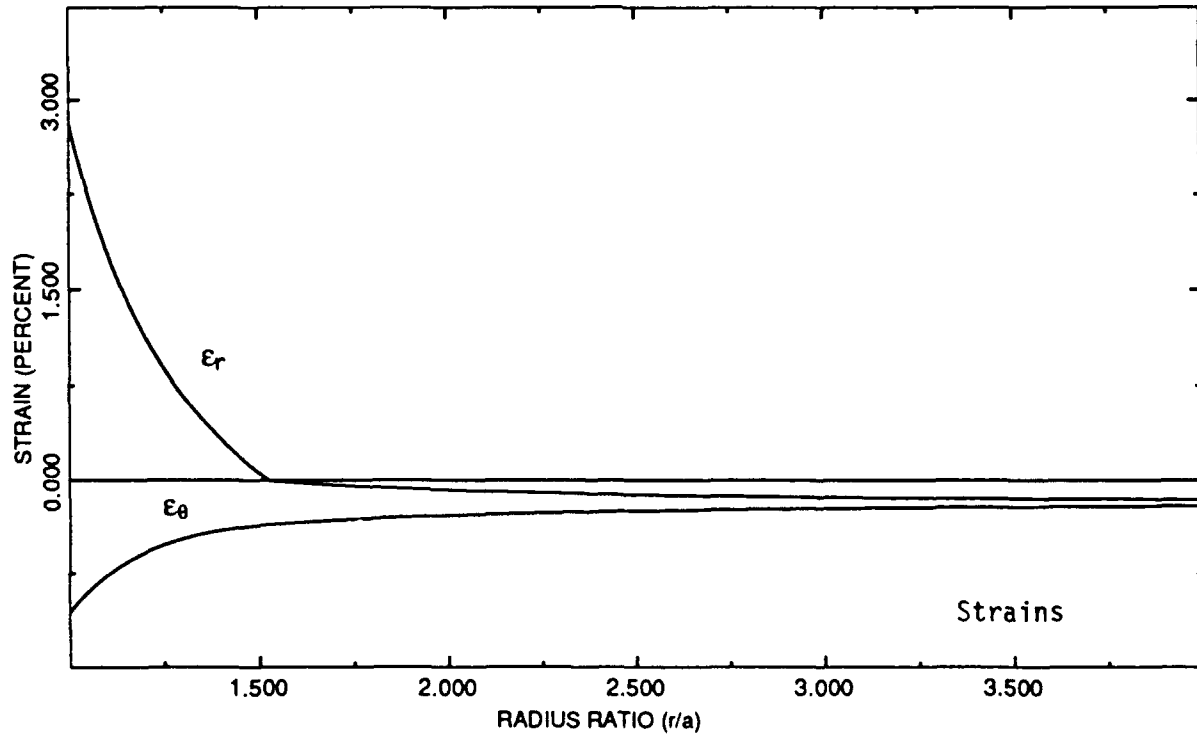
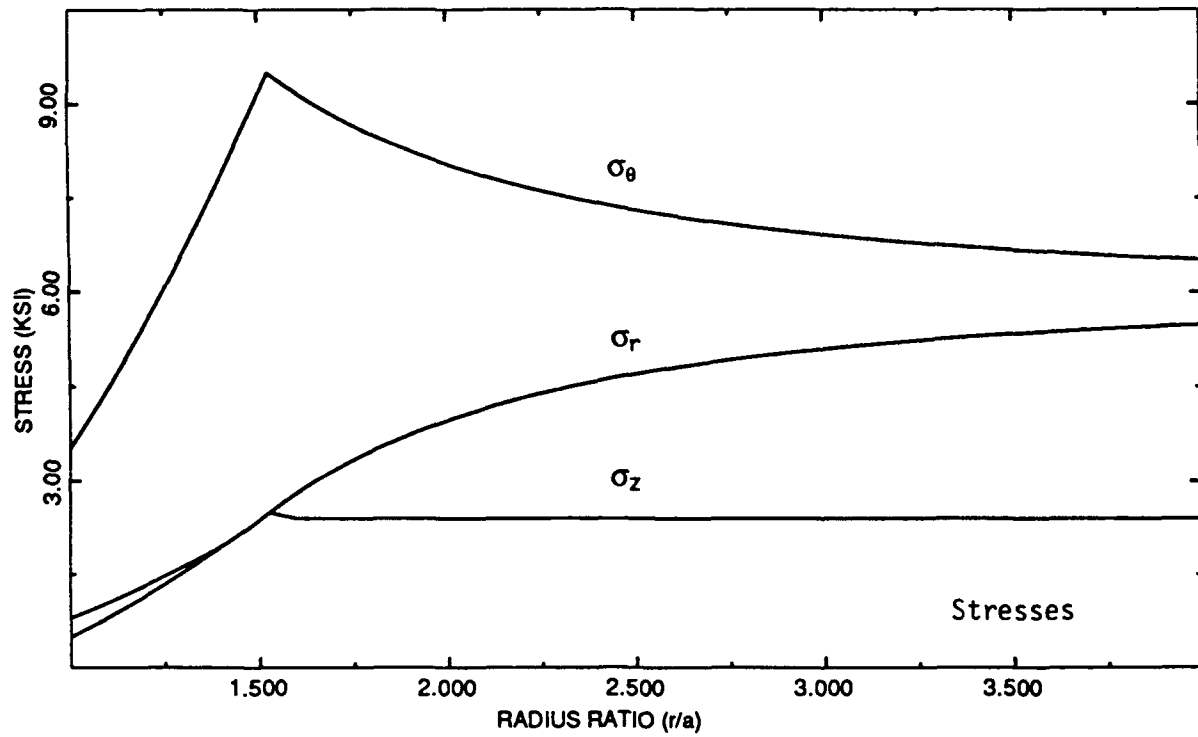


Figure 4. Computed stresses and strains for Case I example with $M = N$.

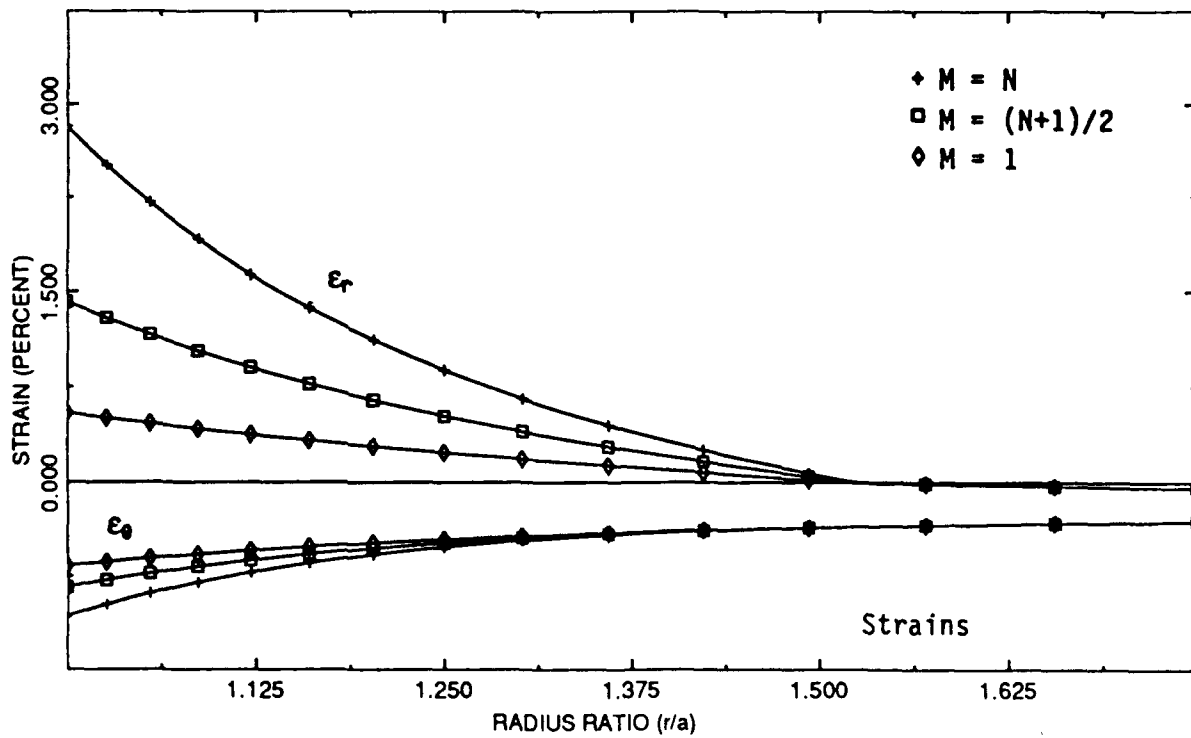
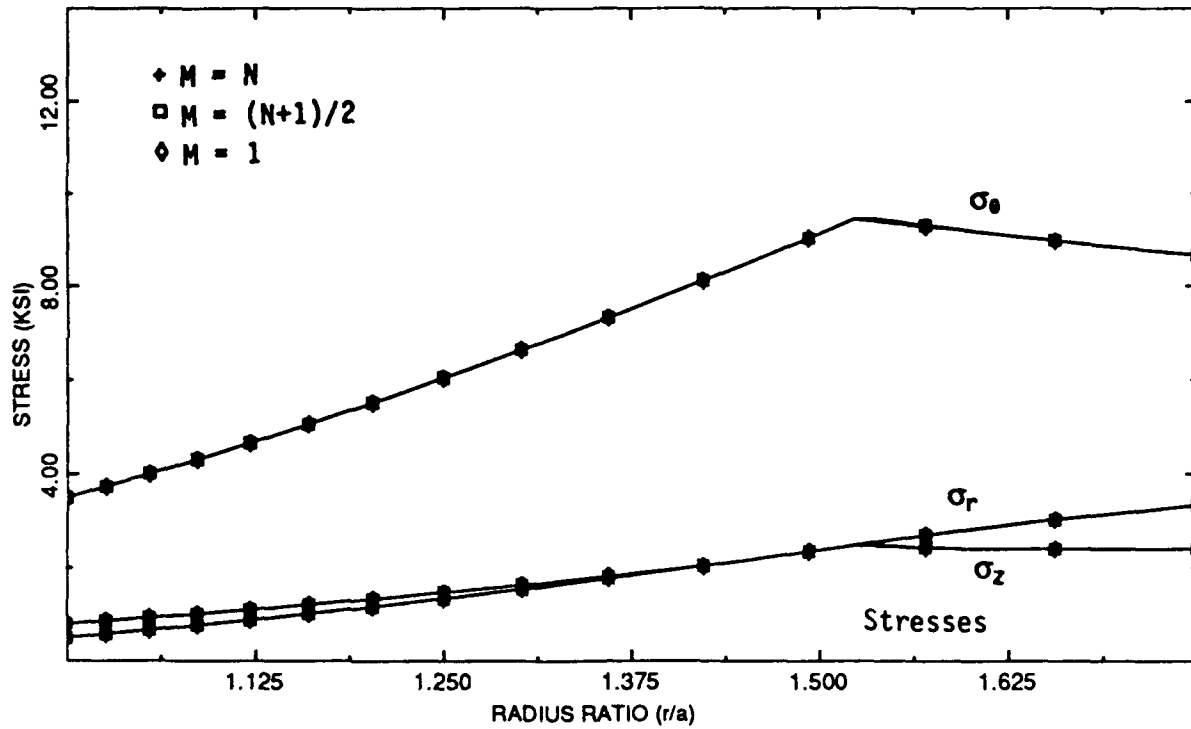


Figure 5. Computed stresses and strains for Case I example with different values of M .

(1) $M = N$, (2) $M = 1$, and $M = (N + 1)/2$.

Beginning at the top of Table 4, $Nv = 0.4 < \frac{1}{2}$. However,

$$\frac{\sigma_u}{(1-2Nv)} = 15,000 > p_s = 2,000$$

so the solution procedure is valid. The next condition is $(N+1)v = 0.6$. Therefore, $(N+1)v < 1$. In addition,

$$\frac{v\sigma_u}{1-(N+1)v} = 1,500$$

and $p_s = 2,000 > 1,500$, which indicates that the Case II yield condition applies, so the procedures of Table 8 should be used.

The first step in Table 8 is to determine whether the specified value of p_b will cause yielding to occur. Therefore,

$$p_b = 6,000 > \frac{p_s + \sigma_u}{2(1-Nv)} = 4,167$$

which indicates that yielding occurs.

Next it is necessary to determine whether one or two plastic zones will form.

In the case of $M = N$,

$$p'_b = 4,529 < p_b = 6,000$$

so the two-plastic-zone procedure applies.

For the upper-bound case where $M = N$, calculated radius ratios are (1) $R'/a = 1.100$, (2) $\bar{R}/a = 1.190$, and (3) $R/a = 1.555$. Stresses and strains for this case are plotted in Figure 6. Except for differences of scale, these plots are identical to the Florence and Schwer results (Figures 2 and 3 of Reference 4). Florence and Schwer state that they used a shear modulus $G = 10^6$ psi (6,895 MPa). However, their plotted strains are consistent with $2G = 10^6$ psi (6,895 MPa).

The two-plastic-zone procedure also applies for the other two values

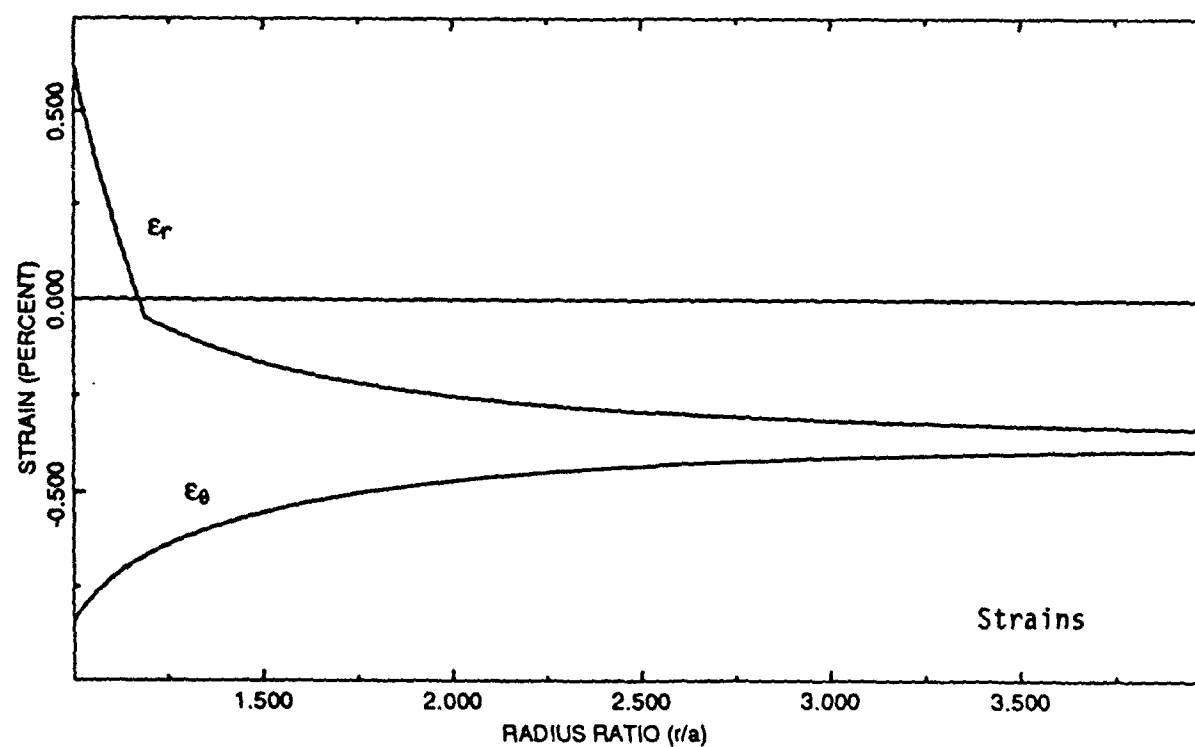
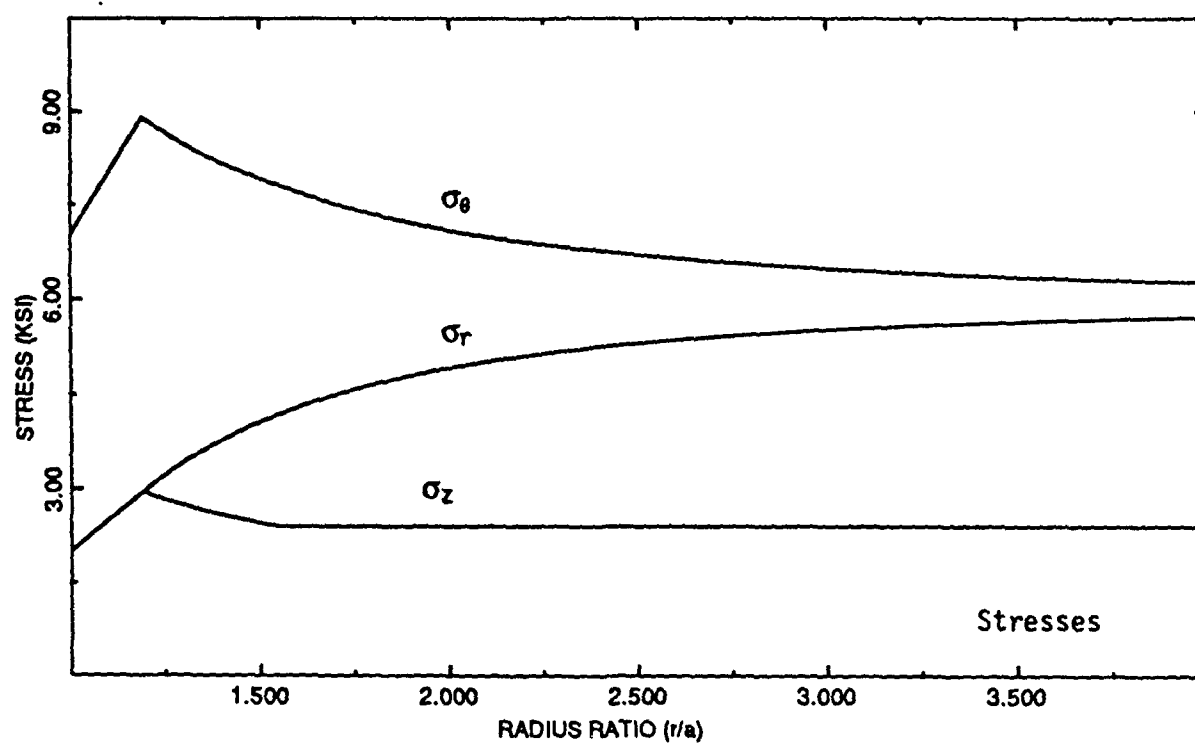


Figure 6. Computed stresses and strains for Case II example with $M = N$.

of M . For the lower bound case where $M = 1$, the calculated radius ratios are (1) $R'/a = 1.114$, (2) $\bar{R}/a = 1.175$, and (3) $R/a = 1.566$. For the intermediate case where $M = (N+1)/2 = 1.5$, the ratios are (1) $R'/a = 1.104$, (2) $\bar{R}/a = 1.185$, and (3) $R/a = 1.558$. Changes in the radius ratios are quite small. R'/a and R/a decrease by 1.3 and 0.7 percent respectively while \bar{R}/a increases by 1.3 percent as M is increased from $M = 1$ to $M = N$.

Stresses and strains for all three cases are plotted in Figure 7. Again, note that the horizontal scale is more than double that of Figure 6. Stresses vary only slightly with changes in M . Radial strains differ significantly at the tunnel wall, but circumferential strains differ only slightly. Both stresses and strains are nearly identical, regardless of the value of M , beyond r/a of about 1.5.

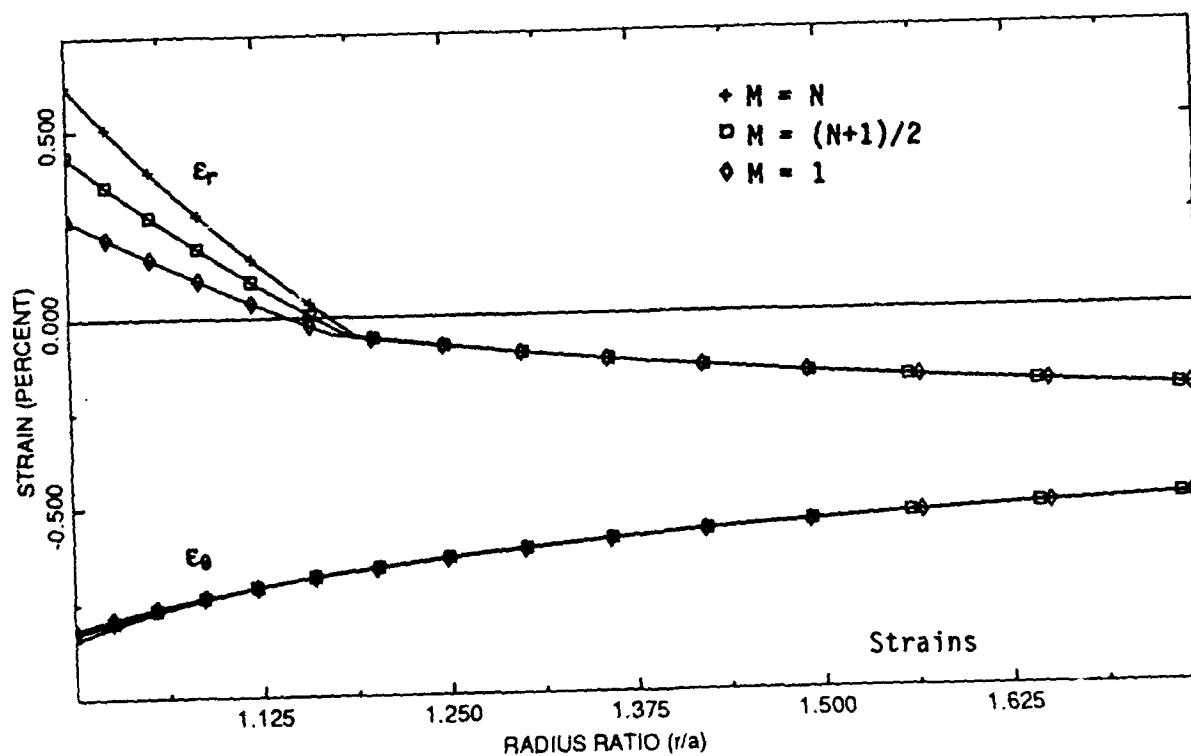
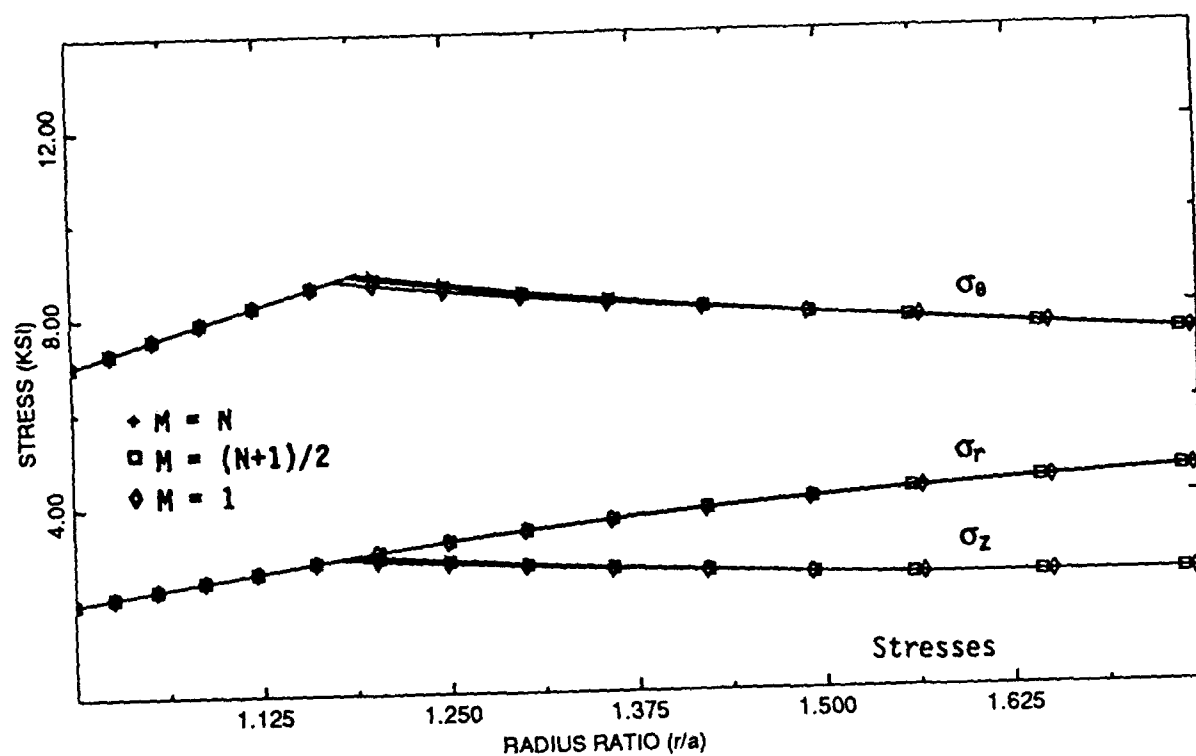


Figure 7. Computed stresses and strains for Case II numerical example with different values of M .

Table 5. Procedure for elastic conditions.

The strain and stress fields are given by

$$2G\epsilon_r = -(1-2\nu)p_b + (p_b - p_a)\frac{a^2}{r^2} \quad (6)$$

$$2G\epsilon_\theta = -(1-2\nu)p_b - (p_b - p_a)\frac{a^2}{r^2} \quad (6)$$

$$\sigma_r = -p_b + (p_b - p_a)\frac{a^2}{r^2} \quad (5)$$

$$\sigma_\theta = -p_b - (p_b - p_a)\frac{a^2}{r^2} \quad (5)$$

$$\sigma_z = -2\nu p_b \quad (5)$$

TUNNEL CLOSURE

$$\frac{\Delta D}{D} = -\epsilon_\theta(a) = -\frac{1}{2G}[p_a - 2p_b(1-\nu)] \quad (6')$$

Table 6. Procedure for Case I yielding--single plastic zone.

For low values of p_b ($p_b \leq \frac{1}{2}[(N+1)p_a + \sigma_u]$), use the procedure for elastic conditions of Table 5. Otherwise:

Compute R

$$\left(\frac{R}{a}\right)^{N-1} = \frac{2}{N+1} \cdot \frac{(N-1)p_b + \sigma_u}{(N-1)p_a + \sigma_u} \quad (34)$$

then $R = a\left(\frac{R}{a}\right)$.

Plastic Zone $a \leq r \leq R$

$$\sigma_r = - \left[p_a + \frac{\sigma_u}{N-1} \right] \left(\frac{r}{a} \right)^{N-1} + \frac{\sigma_u}{N-1} \quad (14)$$

$$2G\epsilon_\theta^{(e)} = [N - (N+1)v]\sigma_r - (1-v)\sigma_u \quad (26)$$

$$2G\epsilon_\theta^{(p)} = -(1-v) \frac{N+1}{M+N} [(N-1)p_a + \sigma_u] \left(\frac{R}{a} \right)^{N-1} \left[\left(\frac{R}{r} \right)^{N+1} - \left(\frac{r}{R} \right)^{N-1} \right] \quad (30')$$

$$2G\epsilon_\theta = 2G\epsilon_\theta^{(e)} + 2G\epsilon_\theta^{(p)}$$

$$2G\epsilon_r = [1 - (N+1)v]\sigma_r + v\sigma_u - M2G\epsilon_\theta^{(p)} \quad (25,17)$$

$$\sigma_\theta = N\sigma_r - \sigma_u \quad (7')$$

$$\sigma_z = v(\sigma_r + \sigma_\theta) \quad (24)$$

p_a^* is required in the elastic zone. Therefore,

$$\sigma_r(R) = - \frac{2p_b - \sigma_u}{N+1} \quad (33)$$

$$p_a^* = -\sigma_r(R)$$

Table 6. Procedure for Case I yielding--single plastic zone
(Concluded).

Elastic Zone $r \geq R$

$$2G\epsilon_r = -(1-2\nu)p_b + (p_b - p_a^*) \frac{R^2}{r^2} \quad (6')$$

$$2G\epsilon_\theta = -(1-2\nu)p_b - (p_b - p_a^*) \frac{R^2}{r^2} \quad (6')$$

$$\sigma_r = -p_b + (p_b - p_a^*) \frac{R^2}{r^2} \quad (5')$$

$$\sigma_\theta = -p_b - (p_b - p_a^*) \frac{R^2}{r^2} \quad (5')$$

$$\sigma_z = -2\nu p_b \quad (5)$$

TUNNEL CLOSURE

$$\begin{aligned} \frac{\Delta D}{D} = -\epsilon_\theta(a) = & -\frac{1}{2G} \left[\{N - (N+1)\nu\} p_a - (1-\nu)\sigma_u \right] \\ & + \frac{1}{2G} \left[(1-\nu) \frac{N+1}{M+N} \{ (N-1)p_a + \sigma_u \} \left(\frac{R}{a} \right)^{N-1} \right] \left[\left(\frac{R}{a} \right)^{N+1} - \left(\frac{a}{R} \right)^{N-1} \right] \end{aligned} \quad (26, 30')$$

Table 7. Procedure for Case I yielding--three plastic zones.

For low values of p_b ($p_b \leq \frac{1}{2}[(N+1)p_a + \sigma_u]$), use the procedure for elastic conditions of Table 5. Next, determine whether there will be one plastic zone or three. If $p_b < \frac{\sigma_u}{2[1-(N+1)v]}$, only a single plastic zone will form and the procedure of Table 6 may be used. Otherwise:

Compute required constants

$$\beta^2 = \frac{MN}{MN+1-(M+N)v} \quad (45)$$

$$\gamma_1 = \beta^2 \frac{(N-M)v}{2MN} + \beta^2 \sqrt{\frac{(N-M)^2 v^2}{(MN)^2} + \frac{1}{\beta^2}} \quad (49)$$

$$\gamma_2 = \beta^2 \frac{(M-N)v}{2MN} + \beta^2 \sqrt{\frac{(M-N)^2 v^2}{(MN)^2} + \frac{1}{\beta^2}} \quad (49)$$

$$A = -\frac{(1-2v)}{1-2Nv} \sigma_u \quad (49)$$

$$c_1 = \frac{2(N-1)v}{1-2Nv} [\sigma_u - (1-2Nv)p_b] \quad (52)$$

$$b_1 = M(N+1)v + [MN+1-(M+N)v]\gamma_1 \quad (53)$$

$$b_2 = M(N+1)v - [MN+1-(M+N)v]\gamma_2 \quad (53)$$

$$c_2 = 2 \frac{1-Nv}{1-2Nv} [MN+1-(M+1)(N+1)v][\sigma_u - (1-2Nv)p_b] \quad (53)$$

$$\beta_1 = \frac{c_2 - b_2 c_1}{b_1 - b_2}, \quad \beta_2 = \frac{b_1 c_1 - c_2}{b_1 - b_2} \quad (53.1)$$

$$b_3 = M[(N+1)v - 1] + [MN+1-(2M+N+1)v]\gamma_1 \quad (55)$$

$$b_4 = M[(N+1)v - 1] - [MN+1-(2M+N+1)v]\gamma_2 \quad (55)$$

$$c_3 = [MN+1-(M+1)(N+1)v] \quad (55)$$

Table 7. Procedure for Case I yielding--three plastic zones
(Continued).

Now determine radii \tilde{R} , \bar{R} , R

$$\frac{\tilde{R}}{a} = \left[\frac{(1-2\nu)\sigma_u}{\{1-(N+1)\nu\}\{(N-1)p_a + \sigma_u\}} \right]^{\frac{1}{N-1}} \quad (38')$$

Then

$$\tilde{R} = a \left(\frac{\bar{R}}{a} \right)$$

The ratio $\frac{\bar{R}}{R}$ may be determined by iteration. Begin by assuming $\frac{\bar{R}}{R} = 0.9$ and solve

$$\frac{\bar{R}}{R} = \left[\frac{c_3 \frac{1-2\nu}{1-2N\nu} \sigma_u - b_3 B_1 \left(\frac{\bar{R}}{R} \right)^{\gamma_1-1}}{b_4 B_2} \right]^{\frac{1}{-\gamma_2-1}} \quad (56')$$

Compute $\frac{\bar{R}}{a}$ from

$$\left(\frac{\bar{R}}{a} \right)^{N-1} = \frac{-b_1 B_1 \left(\frac{\bar{R}}{R} \right)^{\gamma_1-1} - b_2 B_2 \left(\frac{\bar{R}}{R} \right)^{-\gamma_2-1} + c_3 \frac{N(1-2\nu)}{(N-1)(1-2N\nu)} \sigma_u}{c_3 \left(p_a + \frac{\sigma_u}{N-1} \right)} \quad (62')$$

then $\bar{R} = a \left[\left(\frac{\bar{R}}{a} \right)^{N-1} \right]^{\frac{1}{N-1}}$ and $R = \frac{\bar{R}}{\left(\frac{\bar{R}}{R} \right)}$

Compute

$$A_1 = \frac{B_1}{R^{\gamma_1-1}}, \quad A_2 = \frac{B_2}{R^{-\gamma_2-1}} \quad (63)$$

Table 7. Procedure for Case I yielding--three plastic zones
(Continued).

p_a^* is required in the elastic zone. Therefore in the outer plastic zone, compute strains and stresses at $r = R$ (the outer boundary of the zone)

$$2G\epsilon_r(R) = \gamma_1 A_1 R^{\gamma_1-1} - \gamma_2 A_2 R^{-\gamma_2-1} + A \quad (47')$$

$$2G\epsilon_\theta(R) = A_1 R^{\gamma_1-1} + A_2 R^{-\gamma_2-1} + A \quad (48')$$

$$\sigma_r(R) = \frac{1}{C_3} \left[\{MN+1-(M+N)v\} 2G\epsilon_r(R) + M(N+1)v 2G\epsilon_\theta(R) + (M-1)v\sigma_u \right] \quad (42')$$

$$p_a^* = -\sigma_r(R)$$

Inner plastic zone strains depend on circumferential strains at $r = \bar{R}$ and $r = \tilde{R}$ so those are computed next.

$$2G\epsilon_\theta(\bar{R}) = A_1 \bar{R}^{\gamma_1-1} + A_2 \bar{R}^{-\gamma_2-1} + A \quad (48')$$

$$2G\epsilon_\theta(\tilde{R}) = 2G\epsilon_\theta(\bar{R}) \cdot \left(\frac{\tilde{R}}{\bar{R}} \right)^{N+1} + \frac{MN+2-2(M+N+1)v}{(M+N)(N-1)(1+v)} [(N-1)p_a^* + \sigma_u] \left[\left(\frac{\tilde{R}}{\bar{R}} \right)^{N+N} - 1 \right] \left(\frac{\tilde{R}}{a} \right)^{N-1} \\ - \frac{(M+2)(1-2v)\sigma_u}{(N-1)(M+1)(1+v)} \left[\left(\frac{\tilde{R}}{\bar{R}} \right)^{N+1} - 1 \right] \quad (74')$$

The plastic component of $\epsilon_\theta(\tilde{R})$ is also required in the inner plastic zone. It may be determined as follows

$$\sigma_r(\tilde{R}) = -\frac{v\sigma_u}{1-(N+1)v} \quad (35)$$

$$2G\epsilon_\theta^{(*)}(\tilde{R}) = [N-(N+1)v]\sigma_r(\tilde{R}) - (1-v)\sigma_u \quad (26')$$

$$2G\epsilon_\theta^{(p)}(\tilde{R}) = 2G\epsilon_\theta(\tilde{R}) - 2G\epsilon_\theta^{(*)}(\tilde{R}) \quad (19')$$

With the information developed as described above, stresses and strains at any point may be readily determined. For a point at any radius $r \geq a$, compare r to \tilde{R} , \bar{R} , and R to determine its location; then proceed to use the appropriate formulas below.

Table 7. Procedure for Case I yielding--three plastic zones
(Continued).

Inner Plastic Zone ($a \leq r \leq \tilde{R}$)

$$\sigma_r = - \left[p_a + \frac{\sigma_u}{N-1} \right] \left(\frac{r}{a} \right)^{N-1} + \frac{\sigma_u}{N-1} \quad (14)$$

$$\sigma_\theta = N\sigma_r - \sigma_u \quad (7')$$

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) \quad (24)$$

$$2G\epsilon_r^{(e)} = [1 - (N+1)\nu]\sigma_r + \nu\sigma_u \quad (25)$$

$$2G\epsilon_\theta^{(e)} = [N - (N+1)\nu]\sigma_r - (1-\nu)\sigma_u \quad (26)$$

$$2G\epsilon_\theta^{(p)} = 2G\epsilon_\theta^{(p)}(\tilde{R}) \left(\frac{\tilde{R}}{r} \right)^{N+1} - \frac{(N+1)(1-\nu)}{M+N} [(N-1)p_a + \sigma_u] \left(\frac{\tilde{R}}{a} \right)^{N-1} \left[\left(\frac{\tilde{R}}{r} \right)^{N+1} - \left(\frac{r}{\tilde{R}} \right)^{N-1} \right] \quad (76)$$

$$2G\epsilon_\theta = 2G\epsilon_\theta^{(e)} + 2G\epsilon_\theta^{(p)} \quad (19')$$

$$2G\epsilon_r^{(p)} = -M2G\epsilon_\theta^{(p)} \quad (16')$$

$$2G\epsilon_r = 2G\epsilon_r^{(e)} + 2G\epsilon_r^{(p)} \quad (19')$$

Middle Plastic Zone ($\tilde{R} \leq r \leq \bar{R}$)

$$2G\epsilon_\theta = 2G\epsilon_\theta(\bar{R}) \cdot \left(\frac{\bar{R}}{r} \right)^{N+1} + \frac{MN+2-2(M+N+1)\nu}{(M+N)(N-1)(1+\nu)} [(N-1)p_a + \sigma_u] \left[\left(\frac{\bar{R}}{r} \right)^{N+N} - 1 \right] \left(\frac{r}{a} \right)^{N-1} - \frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(M+1)(1+\nu)} \left[\left(\frac{\bar{R}}{r} \right)^{N+1} - 1 \right] \quad (74)$$

$$2G\epsilon_r = -M2G\epsilon_\theta + \frac{MN+2-2(M+N+1)\nu}{1+\nu} \sigma_r - \frac{M-2\nu}{1+\nu} \sigma_u \quad (75)$$

$$\sigma_r = - \left[p_a + \frac{\sigma_u}{N-1} \right] \left(\frac{r}{a} \right)^{N-1} + \frac{\sigma_u}{N-1} \quad (14)$$

$$\sigma_\theta = N\sigma_r - \sigma_u \quad (7')$$

$$\sigma_z = \sigma_r$$

Table 7. Procedure for Case I yielding--three plastic zones
(Continued).

Outer Plastic Zone ($\bar{R} \leq r \leq R$)

$$2G\epsilon_r = \gamma_1 A_1 r^{\gamma_1-1} - \gamma_2 A_2 r^{-\gamma_2-1} + A \quad (47)$$

$$2G\epsilon_\theta = A_1 r^{\gamma_1-1} + A_2 r^{-\gamma_2-1} + A \quad (48)$$

$$\sigma_r = \frac{1}{c_3} [\{MN+1-(M+N)v\}2G\epsilon_r + M(N+1)v2G\epsilon_\theta + (M-1)v\sigma_u] \quad (42')$$

$$\sigma_\theta = \frac{1}{c_3} [N(M+1)v2G\epsilon_r + MN2G\epsilon_\theta - \{1-(M+1)v\}\sigma_u] \quad (43')$$

$$\sigma_z = \frac{1}{N} (\sigma_\theta + \sigma_u) \quad (1')$$

Elastic Zone ($r \geq R$)

$$2G\epsilon_r = -(1-2\nu)p_b + (p_b - p_a^*) \frac{R^2}{r^2} \quad (6')$$

$$2G\epsilon_\theta = -(1-2\nu)p_b - (p_b - p_a^*) \frac{R^2}{r^2} \quad (6')$$

$$\sigma_r = -p_b + (p_b - p_a^*) \frac{R^2}{r^2} \quad (5')$$

$$\sigma_\theta = -p_b - (p_b - p_a^*) \frac{R^2}{r^2} \quad (5')$$

$$\sigma_z = -2\nu p_b \quad (5)$$

Table 7. Procedure for Case I yielding--three plastic zones.
(Concluded).

TUNNEL CLOSURE

Compute constants β^2 , γ_1 , γ_2 , A , b_1 , b_2 , b_3 , b_4 , c_1 , c_2 , c_3 , B_1 , and B_2 as shown at the beginning of the table. Then compute \bar{R} , \bar{R} , and R , followed by A_1 and A_2 .

Now compute tangential strain components as follows.

$$2G\epsilon_\theta(\bar{R}) = A_1 \bar{R}^{\gamma_1-1} + A_2 \bar{R}^{\gamma_2-1} + A \quad (48')$$

$$2G\epsilon_\theta(\tilde{R}) = 2G\epsilon_\theta(\bar{R}) \cdot \left(\frac{\tilde{R}}{\bar{R}}\right)^{M+1} + \frac{MN+2-2(M+N+1)v}{(M+N)(N-1)(1+v)} [(N-1)p_a + \sigma_u] \left[\left(\frac{\tilde{R}}{\bar{R}}\right)^{M+N} - 1\right] \left(\frac{\tilde{R}}{a}\right)^{N-1} \\ - \frac{(M+2)(1-2v)\sigma_u}{(N-1)(M+1)(1+v)} \left[\left(\frac{\tilde{R}}{\bar{R}}\right)^{M+1} - 1\right] \quad (74')$$

$$2G\epsilon_\theta^{(*)}(\bar{R}) = -\frac{[N-(N+1)v]}{1-(N+1)v} \sigma_u - (1-v)\sigma_u \quad (26')$$

$$2G\epsilon_\theta^{(p)}(\bar{R}) = 2G\epsilon_\theta(\bar{R}) - 2G\epsilon_\theta^{(*)}(\bar{R}) \quad (19')$$

$$2G\epsilon_\theta^{(*)}(a) = -[N-(N+1)v]p_a - (1-v)\sigma_u \quad (26')$$

$$2G\epsilon_\theta^{(p)}(a) = 2G\epsilon_\theta^{(p)}(\tilde{R}) \left(\frac{\tilde{R}}{a}\right)^{M+1} - \frac{(N+1)(1-v)}{M+N} [(N-1)p_a + \sigma_u] \left(\frac{\tilde{R}}{a}\right)^{N-1} \left[\left(\frac{\tilde{R}}{a}\right)^{M+1} - \left(\frac{a}{\tilde{R}}\right)^{N-1}\right] \quad (76')$$

Then

$$\frac{\Delta D}{D} = -\epsilon_\theta(a) = -\frac{1}{2G} [2G\epsilon_\theta^{(*)}(a) + 2G\epsilon_\theta^{(p)}(a)] \quad (19')$$

Table 8. Procedure for Case II yielding.

For low values of p_b ($p_b \leq \frac{p_s + \sigma_u}{2(1-Nv)}$), use the procedure for elastic conditions of Table 5.

Determination of the number of plastic zones is not as straight forward as in Case I yielding because calculation of p'_b from Equation 89 is required. It is recommended that all of the following constants be computed first since most of them are required in the calculation of p'_b .

$$\beta^2 = \frac{MN}{MN+1-(M+N)v} \quad (45)$$

$$\gamma_1 = \beta^2 \frac{(N-M)v}{2MN} + \beta^2 \sqrt{\frac{(N-M)^2 v^2}{(MN)^2} + \frac{1}{\beta^2}} \quad (49)$$

$$\gamma_2 = \beta^2 \frac{(M-N)v}{2MN} + \beta^2 \sqrt{\frac{(M-N)^2 v^2}{(MN)^2} + \frac{1}{\beta^2}} \quad (49)$$

$$A = -\frac{(1-2v)}{1-2Nv} \sigma_u \quad (49)$$

$$c_1 = \frac{2(N-1)v}{1-2Nv} [\sigma_u - (1-2Nv)p_b] \quad (52)$$

$$b_1 = M(N+1)v + [MN+1-(M+N)v]\gamma_1 \quad (53)$$

$$b_2 = M(N+1)v - [MN+1-(M+N)v]\gamma_2 \quad (53)$$

$$b_3 = M[(N+1)v-1] + [MN+1-(2M+N+1)v]\gamma_1 \quad (55)$$

$$b_4 = M[(N+1)v-1] - [MN+1-(2M+N+1)v]\gamma_2 \quad (55)$$

$$c_3 = [MN+1-(M+1)(N+1)v] \quad (55)$$

$$c_2 = 2 \frac{1-Nv}{1-2Nv} [MN+1-(M+1)(N+1)v][\sigma_u - (1-2Nv)p_b] \quad (53)$$

$$B_1 = \frac{c_2 - b_2 c_1}{b_1 - b_2}, \quad B_2 = \frac{b_1 c_1 - c_2}{b_1 - b_2} \quad (53.1)$$

Table 8. Procedure for Case II yielding.
(Continued).

$$B_3 = \frac{B_1}{c_1} \quad (85')$$

$$B_4 = \frac{B_2}{c_1} \quad (86')$$

Now determine whether one or two plastic zones will form by computing the ratio a/R' and then p'_b

$$\frac{a}{R'} = \left[\frac{B_3 \cdot \{2v\sigma_u - (1-2Nv)p_a\}b_1 - \{\sigma_u - (1-2Nv)p_a\}\{M + (M+1)v\gamma_1\}}{B_4 \cdot \{2v\sigma_u - (1-2Nv)p_a\}b_2 - \{\sigma_u - (1-2Nv)p_a\}\{M - (M+1)v\gamma_2\}} \right]^{\frac{1}{-\gamma_1 - \gamma_2}} \quad (88)$$

and

$$p'_b = \frac{\sigma_u}{1-2Nv} - \frac{c_3 \left(\frac{\sigma_u}{1-2Nv} - p_a \right)}{2(N-1)v \left[b_1 B_3 \left(\frac{a}{R'} \right)^{\gamma_1-1} + b_2 B_4 \left(\frac{a}{R'} \right)^{-\gamma_2-1} \right]} \quad (89)$$

If $p_b \leq p'_b$, use the single-plastic-zone procedure immediately below. Otherwise, proceed directly to the section below entitled TWO PLASTIC ZONES.

SINGLE PLASTIC ZONE

The ratio a/R may be determined by iteration. Begin by assuming $a/R = 0.9$ and solve

$$\frac{a}{R} = \left[\frac{c_3 \left(\frac{\sigma_u}{1-2Nv} - p_a \right) - b_1 B_1 \left(\frac{a}{R} \right)^{\gamma_1-1}}{b_2 B_2} \right]^{\frac{1}{-\gamma_2-1}} \quad (78')$$

Then $R = a/(a/R)$. Compute constants

$$A_1 = \frac{B_1}{R^{\gamma_1-1}}, \quad A_2 = \frac{B_2}{R^{-\gamma_2-1}} \quad (63)$$

Table 8. Procedure for Case II yielding
(Continued).

With the information developed as described above, stresses and strains at any point may be readily determined. For a point at any radius $r \geq a$, compare r to R to determine its location; then proceed to use the appropriate formulas below.

Plastic Zone ($a \leq r \leq R$)

$$2G\epsilon_r = \gamma_1 A_1 r^{\gamma_1-1} - \gamma_2 A_2 r^{-\gamma_2-1} + A \quad (47)$$

$$2G\epsilon_\theta = A_1 r^{\gamma_1-1} + A_2 r^{-\gamma_2-1} + A \quad (48)$$

$$\sigma_r = \frac{1}{C_3} [\{MN+1-(M+N)v\}2G\epsilon_r + M(N+1)v2G\epsilon_\theta + (M-1)v\sigma_u] \quad (42')$$

$$\sigma_\theta = \frac{1}{C_3} [N(M+1)v2G\epsilon_r + MN2G\epsilon_\theta - \{1-(M+1)v\}\sigma_u] \quad (43')$$

$$\sigma_z = \frac{1}{N} (\sigma_\theta + \sigma_u) \quad (1')$$

Elastic Zone ($r \geq R$)

First find p_a^*

$$\sigma_r(R) = -2(1-Nv)p_b + \sigma_u \quad (50)$$

$$p_a^* = -\sigma_r(R)$$

$$2G\epsilon_r = -(1-2v)p_b + (p_b - p_a^*) \frac{R^2}{r^2} \quad (6')$$

$$2G\epsilon_\theta = -(1-2v)p_b - (p_b - p_a^*) \frac{R^2}{r^2} \quad (6')$$

$$\sigma_r = -p_b + (p_b - p_a^*) \frac{R^2}{r^2} \quad (5')$$

$$\sigma_\theta = -p_b - (p_b - p_a^*) \frac{R^2}{r^2} \quad (5')$$

$$\sigma_z = -2vp_b \quad (5)$$

Table 8. Procedure for Case II yielding
(Continued).

The tunnel closure equation is provided at the end of this table.

TWO PLASTIC ZONES

Determine the radii \bar{R} and R . The ratio \bar{R}/R may be determined by iteration. Begin by assuming $\bar{R}/R = 0.9$ and solve

$$\frac{\bar{R}}{R} = \left[\frac{c_3 \frac{1-2\nu}{1-2N\nu} \sigma_u - b_3 B_1 \left(\frac{\bar{R}}{R} \right)^{\gamma_1-1}}{b_4 B_2} \right]^{\frac{1}{-\gamma_2-1}} \quad (56')$$

compute \bar{R}/a from

$$\left(\frac{\bar{R}}{a} \right)^{N-1} = \frac{-b_1 B_1 \left(\frac{\bar{R}}{R} \right)^{\gamma_1-1} - b_2 B_2 \left(\frac{\bar{R}}{R} \right)^{-\gamma_2-1} + c_3 \frac{N(1-2\nu)}{(N-1)(1-2N\nu)} \sigma_u}{c_3 \left(p_a + \frac{\sigma_u}{N-1} \right)} \quad (62')$$

Then

$$\bar{R} = a \left[\left(\frac{\bar{R}}{a} \right)^{N-1} \right]^{\frac{1}{N-1}} \quad \text{and} \quad R = \frac{\bar{R}}{(\bar{R}/R)}$$

Compute constants

$$A_1 = \frac{B_1}{R^{\gamma_1-1}}, \quad A_2 = \frac{B_2}{R^{-\gamma_2-1}} \quad (63)$$

p_a is required in the elastic zone. Therefore in the outer plastic zone, compute strains and stresses at $r = R$ (the outer boundary of the zone)

$$2G\epsilon_r(R) = \gamma_1 A_1 R^{\gamma_1-1} - \gamma_2 A_2 R^{-\gamma_2-1} + A \quad (47')$$

$$2G\epsilon_\theta(R) = A_1 R^{\gamma_1-1} + A_2 R^{-\gamma_2-1} + A \quad (48')$$

Table 8. Procedure for Case II yielding
(Continued).

$$\sigma_r(R) = \frac{1}{C_3} [\{MN+1-(M+N)v\}2G\epsilon_r(R) + M(N+1)v2G\epsilon_\theta(R) + (M-1)v\sigma_u] \quad (42')$$

$$p_a^* = -\sigma_r(R)$$

Inner Plastic Zone $(r \leq \bar{R})$

$$2G\epsilon_\theta(\bar{R}) = A_1 \bar{R}^{\gamma_1-1} + A_2 \bar{R}^{-\gamma_2-1} + A \quad (48')$$

$$2G\epsilon_\theta = 2G\epsilon_\theta(\bar{R}) \cdot \left(\frac{\bar{R}}{r}\right)^{N+1} + \frac{MN+2-2(M+N+1)v}{(M+N)(N-1)(1+v)} [(N-1)p_a + \sigma_u] \left[\left(\frac{\bar{R}}{r}\right)^{N+N} - 1\right] \left(\frac{r}{a}\right)^{N-1} - \frac{(M+2)(1-2v)\sigma_u}{(N-1)(M+1)(1+v)} \left[\left(\frac{\bar{R}}{r}\right)^{N+1} - 1\right] \quad (74)$$

$$2G\epsilon_r = -2G\epsilon_\theta + \frac{MN+2-2(M+N+1)v}{1+v} \sigma_r - \frac{M-2v}{1+v} \sigma_u \quad (75)$$

$$\sigma_r = - \left[p_a + \frac{\sigma_u}{N-1} \right] \left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \quad (14)$$

$$\sigma_\theta = N\sigma_r - \sigma_u \quad (7')$$

$$\sigma_z = \sigma_r$$

Outer Plastic Zone $(\bar{R} \leq r \leq R)$

$$2G\epsilon_r = \gamma_1 A_1 r^{\gamma_1-1} - \gamma_2 A_2 r^{-\gamma_2-1} + A \quad (47)$$

$$2G\epsilon_\theta = A_1 r^{\gamma_1-1} + A_2 r^{-\gamma_2-1} + A \quad (48)$$

$$\sigma_r = \frac{1}{C_3} [\{MN+1-(M+N)v\}2G\epsilon_r + M(N+1)v2G\epsilon_\theta + (M-1)v\sigma_u] \quad (42')$$

$$\sigma_\theta = \frac{1}{C_3} [N(M+1)v2G\epsilon_r + MN2G\epsilon_\theta - \{1-(M+1)v\}\sigma_u] \quad (43')$$

$$\sigma_z = \frac{1}{N} (\sigma_\theta + \sigma_u) \quad (1')$$

Table 8. Procedure for Case II yielding
(Continued).

Elastic Zone ($r \geq R$)

$$2G\epsilon_r = -(1-2\nu)p_b + (p_b - p_a^*) \frac{R^2}{r^2} \quad (6')$$

$$2G\epsilon_\theta = -(1-2\nu)p_b - (p_b - p_a^*) \frac{R^2}{r^2} \quad (6')$$

$$\sigma_r = -p_b + (p_b - p_a^*) \frac{R^2}{r^2} \quad (5')$$

$$\sigma_\theta = -p_b - (p_b - p_a^*) \frac{R^2}{r^2} \quad (5')$$

$$\sigma_z = -2\nu p_b \quad (5)$$

TUNNEL CLOSURE

Compute constants β^2 , γ_1 , γ_2 , A , b_1 , b_2 , b_3 , b_4 , c_1 , c_2 , c_3 , B_1 , B_2 , B_3 , and B_4 as shown at the beginning of the table. Equations 88 and 89 can then be used to determine if one plastic zone or two will form.

For the single-plastic-zone case, a/R is computed from Equation 78' and constants A_1 and A_2 are found from Equations 63. Tunnel closure can then be determined from

$$\frac{\Delta D}{D} = -\epsilon_\theta(a) = -\frac{1}{2G} (A_1 a^{\gamma_1-1} + A_2 a^{-\gamma_2-1} + A) \quad (48')$$

When two plastic zones form, \bar{R}/R and \bar{R}/a are computed from Equations 56' and 62', respectively. Then

$$\bar{R} = a \left[\left(\frac{\bar{R}}{a} \right)^{n-1} \right]^{\frac{1}{n-1}} \quad \text{and} \quad R = \frac{\bar{R}}{(\bar{R}/R)}$$

Table 8. Procedure for Case II yielding
(Concluded).

Constants A_1 and A_2 can then be computed from Equations 63. Then

$$\epsilon_\theta(\bar{R}) = \frac{1}{2G} [A_1 \bar{R}^{\gamma_1-1} + A_2 \bar{R}^{-\gamma_2-1} + A] \quad (48')$$

and

$$\begin{aligned} \frac{\Delta D}{D} = -\epsilon_\theta(a) = & -\epsilon_\theta(\bar{R}) \cdot \left(\frac{\bar{R}}{a}\right)^{M+1} - \frac{MN+2-2(M+N+1)v}{2G(M+N)(N-1)(1+v)} [(N-1)p_a + \sigma_u] \left[\left(\frac{\bar{R}}{a}\right)^{M+N} - 1\right] \\ & + \frac{(M+2)(1-2v)\sigma_u}{2G(N-1)(M+1)(1+v)} \left[\left(\frac{\bar{R}}{a}\right)^{M+1} - 1\right] \end{aligned} \quad (74')$$

SECTION 6
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APPENDIX A

FLORENCE AND SCHWER CASE I SOLUTION

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AXISYMMETRIC COMPRESSION OF A MOHR-COULOMB MEDIUM AROUND A CIRCULAR HOLE

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SUMMARY

An analytical solution is presented for the stress and strain fields in a Mohr-Coulomb material in plane strain around a circular hole when it is compressed by an axisymmetric far-field pressure. It is shown that several solutions arise involving one to three plastic zones depending on the values of Poisson's ratio and the friction angle. The solution chosen for presentation was obtained and used to validate the functioning of the Mohr-Coulomb yield condition that was added to the NONSAP finite element code. Stress and strain field comparisons are made.

INTRODUCTION

The purpose of this paper is to provide an analytical solution of a rock mechanics problem that can be used to validate the predictions of finite element codes. The problem, which is of interest in itself, is the determination of the stress and strain fields surrounding a tunnel in a Mohr-Coulomb rock subjected far from the tunnel to an axisymmetric pressure. Plane strain conditions prevail and an internal pressure is allowed to develop at the hole boundary. Such a pressure may be considered as arising from a metal liner at yield or from a crushable back-packing material between the liner and the rock. To simplify the analysis and to provide a defined loading history this internal pressure is assumed to be equal to the far-field pressure as the far-field pressure is increased (loading) until the internal pressure reaches its final value, whereupon it remains constant with subsequent loading.

The rock behaviour is elastic-perfectly plastic, obeying the Mohr-Coulomb yield condition and associated flow rule; hence, dilation is included. It will be seen that several cases can arise that depend on the relative magnitudes of the principal stresses, and these relative magnitudes depend on the values of Poisson's ratio and the friction angle. One of these cases is treated here, chosen to involve two adjacent faces and the common edge of the yield pyramid, instead of the possible case involving only one face, to provide a more severe test for the code. Three plastic zones arise, shown in Figure 1, that correspond to the two faces and common edge. In this work the finite element code employed is NONSAP¹ to which the Mohr-Coulomb yield condition presented in Reference 2 has been added. Comparison of analytical and code results is presented.

The motivation for the code validation analysis was to provide support for the experimental and theoretical investigations of Kennedy and Lindberg³ and Senseny and Lindberg⁴ into the response of buried cylindrical structures to quasi-static loading.

Initial loading—outer elastic zone

Let p_a be the pressure acting on the hole boundary of radius a and let p_b be the pressure at infinity (far-field). If $p_a = p_b$ on initial loading, equilibrium requires that the radial and circum-

0363-0961/78/0402-0367\$01.00

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Received 27 September 1977

Revised 15 December 1977

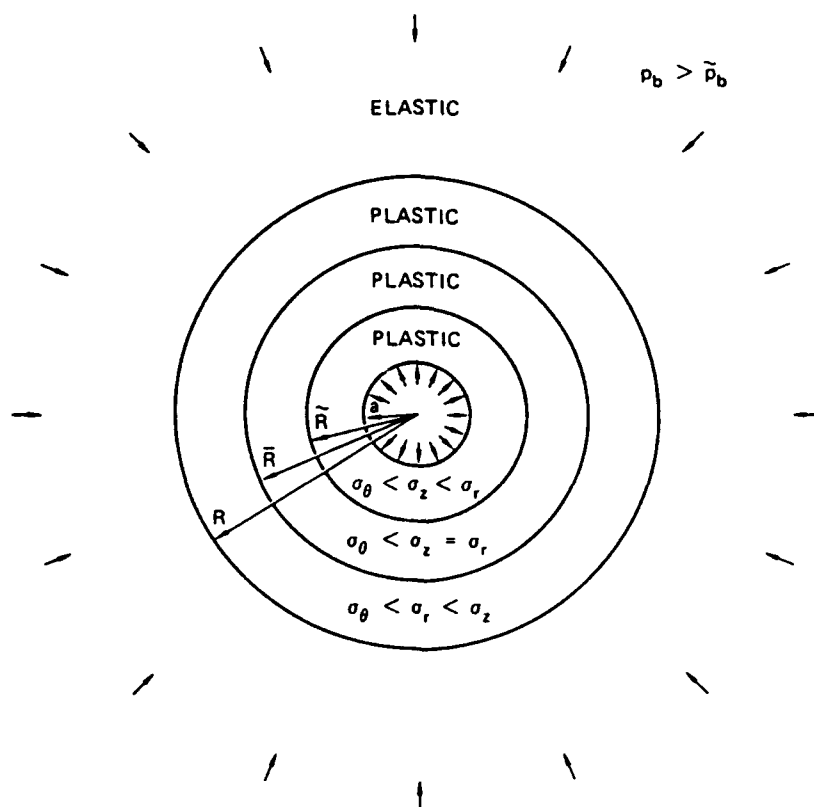


Figure 1. Outer elastic and three plastic zones

ferential stress components are equal, that is, $\sigma_r = \sigma_\theta = -p_b$. If the material is elastic and in a plane strain condition, the axial stress is $\sigma_z = -2\nu p_b$ throughout, where ν is Poisson's ratio ($0 < \nu < \frac{1}{2}$). Hence, we have the relationship $\sigma_\theta = \sigma_r < \sigma_z$. With increased loading, yielding will occur if

$$f = \sigma_\theta - N\sigma_z + \sigma_u = 0 \quad (1)$$

where

$$N = \frac{1 + \sin \varphi}{1 - \sin \varphi} \quad (2)$$

and $\sigma_u > 0$ the unconfirmed crush strength φ being the friction angle. Substitution of the stress components in yield condition (1) gives

$$p_a = p_b = \frac{\sigma_u}{1 - 2N\nu} \quad (3)$$

For $p_b > 0$, the material will not yield if $2N\nu > 1$. However, if $2N\nu < 1$ the material will yield throughout when equation (3) is satisfied. We shall therefore consider design values of internal

pressure p_a obeying the inequality

$$p_a < \frac{\sigma_u}{1-2N\nu} \quad \text{when } 2N\nu < 1 \quad (4)$$

and for p_a , so far, unrestricted when $2N\nu > 1$.

Once the design internal pressure has been reached it is held constant and the far-field pressure is increased. While the material remains elastic the stress and strain fields about a hole of radius $r = a$ are

$$\sigma_r = -p_b + (p_b - p_a)\frac{a^2}{r^2}, \quad \sigma_\theta = -p_b - (p_b - p_a)\frac{a^2}{r^2}, \quad \sigma_z = 2\nu p_b \quad (5)$$

$$2G\epsilon_r = -(1-2\nu)p_b + (p_b - p_a)\frac{a^2}{r^2}, \quad 2G\epsilon_\theta = -(1-2\nu)p_b - (p_b - p_a)\frac{a^2}{r^2}, \quad \epsilon_z = 0 \quad (6)$$

where, for our loading history, $p_a < p_b$. In (5), G is the shear modulus. We see immediately from (5) that $\sigma_\theta < \sigma_r$ and $\sigma_\theta < \sigma_z$ throughout. Also, $\sigma_r(\infty) < \sigma_z$, but at the hole the inequalities are conditional, that is, $\sigma_r(a) > \sigma_z$ if $p_a < 2\nu p_b$ and $\sigma_z < \sigma_r(a)$ if $p_a < 2\nu p_b$. It can readily be shown that with increasing loading, yielding occurs first at the hole. We now examine the effect these inequalities have on the initial yielding at the hole.

Initial yielding

Let yielding occur at $r = a$ when $p_b = \bar{p}_b$. If $p_a < 2\nu\bar{p}_b$ when yielding occurs, then $\sigma_\theta < \sigma_z < \sigma_r$ at $r = a$ and the yield condition is

$$f = \sigma_\theta - N\sigma_r + \sigma_u = 0 \quad (7)$$

Substitution of $\sigma_\theta(a) = -2\bar{p}_b + p_a$ and $\sigma_r(a) = -p_a$, from (5), in equation (7) leads to the yield pressure

$$\bar{p}_b = \frac{1}{2}[(N+1)p_a + \sigma_u] \quad (8)$$

Satisfaction of the inequality $p_a < 2\nu\bar{p}_b$ now requires

$$p_a < \frac{\nu\sigma_u}{1-(N+1)\nu} \quad (N+1)\nu < 1 \quad (9)$$

but places no restriction on positive p_a when $(N+1)\nu > 1$.

If $2\nu\bar{p}_b < p_a < \bar{p}_b$ when yielding occurs, then $\sigma_\theta < \sigma_r < \sigma_z$ at $r = a$ and the yield condition is equation (1). Substitution of $\sigma_\theta(a) = -2\bar{p}_b + p_a$ and $\sigma_z(a) = -2\nu\bar{p}_b$ in equation (1) leads to the yield pressure

$$\bar{p}_b = \frac{p_a + \sigma_u}{2(1-N\nu)} \quad N\nu < 1 \quad (10)$$

Yield condition (1) does not govern yielding if $N\nu > 1$. In this case, p_b increases until $p_a < 2\nu\bar{p}_b$ so that yielding is governed by condition (7) and \bar{p}_b is given by formula (8); because $(N+1)\nu > 1$, no restriction is placed on p_a . For $N\nu < (N+1)\nu < 1$, inequality $2\nu\bar{p}_b < p_a < \bar{p}_b$

and the yield pressure (10) restrict p_a to the ranges

$$\frac{\nu\sigma_u}{1-(N+1)\nu} < p_a < \frac{\sigma_u}{1-2N\nu} \quad 0 < N\nu < \frac{1}{2} \quad (11)$$

$$\frac{\nu\sigma_u}{1-(N+1)\nu} < p_a \quad \frac{1}{2} < N\nu < (N+1)\nu < 1 \quad (12)$$

The above results are collected in Table I.

Table I. Initial yielding conditions

Property relations	Internal pressure	Yield pressure	Stress order	Yielding case
$N\nu < (N+1)\nu < 1$	$p_a < \frac{\nu\sigma_u}{1-(N+1)\nu}$	$\bar{p}_b = \frac{1}{2}[(N+1)p_a + \sigma_u]$	$\sigma_\theta < \sigma_z < \sigma_r$	(1) $p_a < 2\nu\bar{p}_b$
$\left. \begin{array}{l} 1 < N\nu < (N+1)\nu \\ N\nu < 1 < (N+1)\nu \end{array} \right\}$	$p_a > 0$			
$\frac{1}{2} < N\nu < (N+1)\nu < 1$	$p_a > \frac{\nu\sigma_u}{1-(N+1)\nu}$	$\bar{p}_b = \frac{p_a + \sigma_u}{2(1-N\nu)}$	$\sigma_\theta < \sigma_r < \sigma_z$	(2) $2\nu\bar{p}_b < p_a < \bar{p}_b$
$0 < N\nu < \frac{1}{2}$	$\frac{\nu\sigma_u}{1-(N+1)\nu} < p_a < \frac{\sigma_u}{1-2N\nu}$			

We present here the analysis of Case 1 in Table I for $p_b > \bar{p}_b$, where \bar{p}_b is given by (8). In order to avoid an excessively long exposition, Case 2 of Table I will be presented in a forthcoming paper.

Initial growth of plastic zone

As the far-field pressure is increased beyond the initial yield value \bar{p}_b , an annular plastic zone of outer radius $r = R$ expands from the hole of radius $r = a$. within this zone yielding is governed by condition (7). the equation of equilibrium throughout the material is

$$r \frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = 0 \quad (13)$$

After eliminating σ_θ from (13) by means of yield condition (7) it is found that the solution of the resulting equation that satisfies $\sigma_r(a) = -p_a$ is

$$\sigma_r = - \left[p_a + \frac{\sigma_u}{N-1} \right] \left(\frac{r}{a} \right)^{N-1} + \frac{\sigma_u}{N-1} \quad a < r < R \quad (14)$$

The flow rule associated with yield condition (7) is

$$\dot{\epsilon}_r^{(p)} = \lambda \frac{\partial f}{\partial \sigma_r} = -\lambda N, \quad \dot{\epsilon}_\theta^{(p)} = \lambda \frac{\partial f}{\partial \sigma_\theta} = \lambda, \quad \dot{\epsilon}_z^{(p)} = \lambda \frac{\partial f}{\partial \sigma_z} = 0 \quad (15)$$

where the dot may be interpreted as differentiation with respect to p_b . From flow rule (15) it follows that

$$[\dot{\epsilon}_r^{(p)} + N\dot{\epsilon}_\theta^{(p)}] = 0, \quad \dot{\epsilon}_r^{(p)} + N\dot{\epsilon}_\theta^{(p)} = g_1(r), \quad \dot{\epsilon}_z^{(p)} = g_2(r) \quad (16)$$

which means that the strain expressions in (16) evaluated at a fixed radius r remain constant as loading proceeds. Initially, when the elastic-plastic radius R is at this radius r , the plastic strains are zero so that $g_1(r) = 0$ and $g_2(r) = 0$. Hence,

$$\dot{\epsilon}_r^{(p)} + N\dot{\epsilon}_\theta^{(p)} = 0, \quad \dot{\epsilon}_z^{(p)} = 0 \quad (17)$$

Strain increments are taken as the sum of the elastic and plastic strain increments, that is,

$$\dot{\epsilon}_r = \dot{\epsilon}_r^{(e)} + \dot{\epsilon}_r^{(p)}, \quad \dot{\epsilon}_\theta = \dot{\epsilon}_\theta^{(e)} + \dot{\epsilon}_\theta^{(p)}, \quad \dot{\epsilon}_z = \dot{\epsilon}_z^{(e)} + \dot{\epsilon}_z^{(p)} \quad (18)$$

Now $\epsilon_r - (\epsilon_r^{(e)} + \epsilon_r^{(p)}) = g(r)$, and when $R = r$ we have $\epsilon_r = \epsilon_r^{(e)}$ and $\epsilon_r^{(p)} = 0$ so that $g(r) = 0$. Using the same argument for the other two strain components leads to

$$\epsilon_r = \epsilon_r^{(e)} + \epsilon_r^{(p)}, \quad \epsilon_\theta = \epsilon_\theta^{(e)} + \epsilon_\theta^{(p)}, \quad \epsilon_z = \epsilon_z^{(e)} + \epsilon_z^{(p)} \quad (19)$$

Plane strain requires $\epsilon_z = 0$ and the flow rule leads to $\epsilon_z^{(p)} = 0$, as shown in (17). Thus (19) for the sum of the strains completes the deduction that

$$\epsilon_z = 0, \quad \epsilon_z^{(e)} = 0, \quad \epsilon_z^{(p)} = 0 \quad (20)$$

Hooke's law relating the elastic strains to the stresses is

$$E\epsilon_r^{(e)} = \sigma_r - \nu(\sigma_\theta + \sigma_z) \quad (21)$$

$$E\epsilon_\theta^{(e)} = \sigma_\theta - \nu(\sigma_z + \sigma_r) \quad (22)$$

$$E\epsilon_z^{(e)} = \sigma_z - \nu(\sigma_r + \sigma_\theta) \quad (23)$$

where E is Young's modulus. The result $\epsilon_z^{(e)} = 0$ substituted in (23) gives

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) \quad (24)$$

Elimination of σ_θ and σ_z from (21) and (22) by using yield condition (7) and result (24) leads to

$$2G\epsilon_r^{(e)} = [1 - (N+1)\nu]\sigma_r + \nu\sigma_u \quad (25)$$

$$2G\epsilon_\theta^{(e)} = [N - (N+1)\nu]\sigma_r - (1-\nu)\sigma_u \quad (26)$$

where G is the shear modulus and σ_r is given by (14).

The strains are related to the radial displacement u by

$$\epsilon_r = \frac{du}{dr}, \quad \epsilon_\theta = \frac{u}{r} \quad (27)$$

and elimination of u leads to the compatibility equation

$$r \frac{d\epsilon_\theta}{dr} + \epsilon_\theta - \epsilon_r = 0 \quad (28)$$

By using in turn (19) for decomposing the strains, (17) to eliminate $\epsilon_r^{(p)}$, (25) and (26) to eliminate the elastic strains and (14) to eliminate σ_r , equation (28) becomes

$$r \frac{d\epsilon_\theta^{(p)}}{dr} + (N+1)\epsilon_\theta^{(p)} = \frac{1-\nu}{2G} \cdot (N+1)[(N-1)p_a + \sigma_u] \left(\frac{r}{a}\right)^{N-1} \quad (29)$$

The solution of equation (29) satisfying $\epsilon_{\theta}^{(p)}(R) = 0$ is

$$\epsilon_{\theta}^{(p)} = -\frac{1-\nu}{2G} \cdot \frac{N+1}{N} \cdot [N-1]p_a + \sigma_u \left(\frac{R}{a}\right)^{N-1} \left[\left(\frac{R}{r}\right)^{N+1} - \left(\frac{r}{R}\right)^{N-1} \right] \quad (30)$$

and $\epsilon_r^{(p)}$ is given by (17). The remaining unknown is the radius R . Using R and p instead of a and p_a in (5) gives

$$\sigma_r(R) = -p, \quad \sigma_{\theta}(R) = -2p_b + p, \quad \sigma_z(R) = -2\nu p_b \quad (31)$$

If yield condition (7) still holds at $r = R$ it follows that

$$p_b = \frac{1}{2}[N+1]p + \sigma_u \quad (32)$$

$$\sigma_r(R) = -\frac{2p_b - \sigma_u}{N+1} \quad (33)$$

Substituting $r = R$ in formula (14) and equating to formula (33) provides continuity of the radial stress and the result

$$\left(\frac{R}{a}\right)^{N-1} = \frac{2}{N+1} \cdot \frac{(N-1)p_b + \sigma_u}{(N-1)p_a + \sigma_u} \quad (34)$$

The above description of yielding applies with continued loading as long as $\sigma_z < \sigma_r$ in the plastic zone. From (24) and (7),

$$\sigma_r - \sigma_z = [1 - (N+1)\nu]\sigma_r + \nu\sigma_u$$

so that

$$(1) \quad \sigma_z < \sigma_r \quad \text{when } (N+1)\nu > 1$$

$$(2) \quad \sigma_z < \sigma_r \quad \text{when } (N+1)\nu < 1 \quad \text{if } \sigma_r < -\frac{\nu\sigma_u}{1 - (N+1)\nu}$$

In the first case, when $(N+1)\nu > 1$, the solution is complete; this case has been treated in the literature.^{3,5} In the second case, when $(N+1)\nu < 1$, we first observe that in the plastic zone $\min \sigma_r = \sigma_r(R)$ so that as loading proceeds the elastic-plastic radius R attains a value \tilde{R} where

$$\sigma_r(\tilde{R}) = \sigma_z(\tilde{R}) = -\frac{\nu\sigma_u}{1 - (N+1)\nu} \quad (35)$$

and

$$\sigma_{\theta}(\tilde{R}) = -\frac{(1-\nu)\sigma_u}{1 - (N+1)\nu} \quad (36)$$

The magnitude of the loading when $R = \tilde{R}$ is \tilde{p}_b given by (31) and (35), that is,

$$\tilde{p}_b = \frac{\sigma_u}{2[1 - (N+1)\nu]} \quad (37)$$

and the radius of the elastic-plastic interface, from (34) and (37), is determined by

$$\left(\frac{\tilde{R}}{a}\right)^{N-1} = \frac{(1-2\nu)\sigma_u}{[1 - (N+1)\nu][(N-1)p_a + \sigma_u]} \quad (38)$$

Relationship (38) for \bar{R} is independent of \bar{p}_b as is the stress field, and consequently both remain constant during further loading.

We now show that with further loading three plastic zones appear, as shown in Figure 1.

Increased loading—outer plastic zone

It is postulated that with increased loading ($p_b > \bar{p}_b$) plastic yielding outside the fixed radius \bar{R} and adjacent to the outer infinite elastic zone occupies an annular region $\bar{R} < r < R$ in which $\sigma_\theta < \sigma_r < \sigma_z$, as shown in Figure 1. The yield condition in this zone is therefore equation (1) and the associated flow rule is

$$\dot{\epsilon}_r^{(p)} = 0, \quad \dot{\epsilon}_\theta^{(p)} = \lambda, \quad \dot{\epsilon}_z^{(p)} = -N\lambda \quad (39)$$

which, for the same reasoning that led from flow rule (15) to the plastic strain relations (17), provides the strain relations

$$\epsilon_z^{(p)} + N\epsilon_\theta^{(p)} = 0, \quad \epsilon_r^{(p)} = 0 \quad (40)$$

Relations (19), the plane strain condition $\epsilon_z = 0$ and relations (40) lead to the strains

$$\epsilon_r = \epsilon_r^{(e)}, \quad \epsilon_\theta = \epsilon_\theta^{(e)} + \frac{1}{N}\epsilon_z^{(e)}, \quad \epsilon_z = 0 \quad (41)$$

Hooke's law (21)–(23), expressions (41) and yield condition (1) give

$$[N^2 + 1 - (N+1)^2\nu]\sigma_r = (N^2 + 1 - 2N\nu)2G\epsilon_r + N(N+1)\nu 2G\epsilon_\theta + (N-1)\nu\sigma_u \quad (42)$$

$$[N^2 + 1 - (N+1)^2\nu]\sigma_\theta = N(N+1)\nu 2G\epsilon_r + N^2 2G\epsilon_\theta - [1 - (N+1)\nu]\sigma_u \quad (43)$$

Substitution of the stresses (42) and (43) in the equation of equilibrium (13) and use of the strain-displacement relations (27) lead to the displacement equation

$$r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} - \alpha^2 u = -\frac{\alpha^2 (1-2\nu)\sigma_u}{2G} \quad (44)$$

where

$$\alpha^2 = N^2 / (N^2 + 1 - 2N\nu) \quad (45)$$

The displacement solutions of (44) and the corresponding strains are

$$2Gu = A_1 r^\alpha + A_2 r^{-\alpha} + Ar \quad (46)$$

$$2G\epsilon_r = \alpha A_1 r^{\alpha-1} - \alpha A_2 r^{-\alpha-1} + A \quad (47)$$

$$2G\epsilon_\theta = A_1 r^{\alpha-1} + A_2 r^{-\alpha-1} + A \quad (48)$$

where

$$A = -\frac{(1-2\nu)\sigma_u}{1-2N\nu} \quad (49)$$

If the elastic zone stresses evaluated at $r = R$ are substituted in yield condition (1) we find that

$$\sigma_r(R) = -2(1-N\nu)p_b + \sigma_u \quad (50)$$

Hence, the circumferential strain at $r = R$ is given by

$$2G\epsilon_\theta(R) = -2(N-1)\nu p_b - \sigma_u \quad (51)$$

The radial stress and circumferential strain at $r = R$ from the plastic zone are determined by (42), (47) and (48). Thus, to provide continuity of radial stress and displacement (or ϵ_θ) at $r = R$ we require, according to (50) and (51),

$$A_1 R^{\alpha-1} + A_2 R^{-\alpha-1} = \frac{2(N-1)\nu}{1-2N\nu} [\sigma_u - (1-2N\nu)p_b] \quad (52)$$

$$\begin{aligned} \left[\frac{N}{\alpha} + (N+1)\nu \right] A_1 R^{\alpha-1} - \left[\frac{N}{\alpha} - (N+1)\nu \right] A_2 R^{-\alpha-1} \\ = \frac{2(1-N\nu)[N^2+1-(N+1)^2\nu]}{N(1-2N\nu)} [\sigma_u - (1-2N\nu)p_b] \end{aligned} \quad (53)$$

It is reasonable to postulate that at the inner radius \bar{R} of this plastic zone we have $\sigma_r(\bar{R}) = \sigma_z(\bar{R})$ so that the yield condition may be written as

$$N\sigma_r(\bar{R}) - \sigma_\theta(\bar{R}) = \sigma_u \quad (54)$$

where the stresses are given by (42) and (43). Substitution of (42) and (43) in (54) and use of strain solutions (47) and (48) with $r = \bar{R}$ lead to the equation

$$\begin{aligned} \left(\frac{N}{\alpha} - 1 \right) \left[\frac{N}{\alpha} + (N+1)\nu \right] A_1 \bar{R}^{\alpha-1} - \left(\frac{N}{\alpha} + 1 \right) \left[\frac{N}{\alpha} - (N+1)\nu \right] A_2 \bar{R}^{-\alpha-1} \\ = \frac{[N^2+1-(N+1)^2\nu](1-2\nu)\sigma_u}{\alpha(1-2N\nu)} \end{aligned} \quad (55)$$

After solving (52) and (53) for A_1 and A_2 , and substituting in (55) we arrive at the equation

$$\begin{aligned} \left(\frac{N}{\alpha} - 1 \right) \left[\frac{N}{\alpha} + (N+1)\nu \right] \left\{ \left[\frac{N}{\alpha} - (N+1)\nu \right] (N-1)N\nu + [N^2+1-(N+1)^2\nu](1-N\nu) \right\} \left(\frac{\bar{R}}{R} \right)^{\alpha-1} \\ - \left(\frac{N}{\alpha} + 1 \right) \left[\frac{N}{\alpha} - (N+1)\nu \right] \\ \cdot \left\{ \left[\frac{N}{\alpha} + (N+1)\nu \right] (N-1)N\nu - [N^2+1-(N+1)^2\nu](1-N\nu) \right\} \left(\frac{\bar{R}}{R} \right)^{-\alpha-1} \\ = (N^2+1-2N\nu)[N^2+1-(N+1)^2\nu] \frac{(1-2\nu)\sigma_u}{\sigma_u - (1-2N\nu)p_b} \end{aligned} \quad (56)$$

Equation (56) determines the radius ratio \bar{R}/R . Determination of \bar{R} requires a description of the state in the plastic zone $\bar{R} < r < R$.

Increased loading—middle plastic zone

It is postulated that with increased loading ($p_b > \bar{p}_b$) $\sigma_r = \sigma_z$ throughout the plastic zone $\bar{R} < r < R$ shown in Figure 1, so that yielding is governed by

$$f_1 = \sigma_\theta - N\sigma_r + \sigma_u = 0 \quad (57)$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \quad (58)$$

Equation (57) allows the elimination of σ_θ from the equilibrium equation (13) to leave a differential equation with the solution

$$\sigma_r = \sigma_r(\bar{R}) \cdot \left(\frac{r}{\bar{R}} \right)^{N-1} + \left[1 - \left(\frac{r}{\bar{R}} \right)^{N-1} \right] \frac{\sigma_u}{N-1} \quad (59)$$

which, after substituting \bar{R} and $\sigma_r(\bar{R})$ from (38) and (35), becomes

$$\sigma_r = - \left[p_a + \frac{\sigma_u}{N-1} \right] \left(\frac{r}{a} \right)^{N-1} + \frac{\sigma_u}{N-1} \quad (60)$$

Because formulae (60) and (14) are the same, the radial stress σ_r is given by the same formula throughout the combined inner and middle zones, $a < r < \bar{R}$. Formula (60) gives the value $\sigma_r(\bar{R})$, so providing continuity of this stress at the interface $r = \bar{R}$ leads to the equation

$$\begin{aligned} & \left[\frac{N}{\alpha} + (N+1)\nu \right] \left[\left\{ \frac{N}{\alpha} - (N+1)\nu \right\} (N-1)N\nu + \{N^2 + 1 - (N+1)^2\nu\} (1-N\nu) \right] \left(\frac{\bar{R}}{R} \right)^{\alpha-1} \\ & + \left[\frac{N}{\alpha} - (N+1)\nu \right] \left[\left\{ \frac{N}{\alpha} + (N+1)\nu \right\} (N-1)N\nu - \{N^2 + 1 - (N+1)^2\nu\} (1-N\nu) \right] \left(\frac{\bar{R}}{R} \right)^{-\alpha-1} \\ & = \frac{(N^2 + 1 - 2N\nu)[N^2 + 1 - (N+1)^2\nu]}{(N-1)[\sigma_u - (1-2N\nu)p_b]} \left[(1-2\nu)\sigma_u - (1-2N\nu)\{(N-1)p_a + \sigma_u\} \left(\frac{\bar{R}}{a} \right)^{N-1} \right] \end{aligned} \quad (61)$$

The radius ratio \bar{R}/R is determined by (56), so equation (61) determines the radius \bar{R} and hence R is also determined. Now that R is known, the constants A_1 and A_2 (in terms of the loading parameter p_b) are the determinable solutions of (52) and (53). Explicitly,

$$A_1 = R^{1-\alpha} \left[\left\{ \frac{N}{\alpha} - (N+1)\nu \right\} (N-1)N\nu + \{N^2 + 1 - (N+1)^2\nu\} (1-N\nu) \right] \frac{\alpha[\sigma_u - (1-2N\nu)p_b]}{N^2(1-2N\nu)} \quad (62)$$

$$A_2 = R^{1+\alpha} \left[\left\{ \frac{N}{\alpha} + (N+1)\nu \right\} (N-1)N\nu - \{N^2 + 1 - (N+1)^2\nu\} (1-N\nu) \right] \frac{\alpha[\sigma_u - (1-2N\nu)p_b]}{N^2(1-2N\nu)} \quad (63)$$

and by (46), (47) and (48) the displacements and strains in the outer plastic zone are known. Finally, by (42), (43) and (1) the stresses in the outer plastic zone are known.

The flow rule associated with the yield condition equations (57) and (58) is

$$\dot{\epsilon}_r^{(p)} = \lambda_1 \frac{\partial f_1}{\partial \sigma_r} + \lambda_2 \frac{\partial f_2}{\partial \sigma_r} = -N\lambda_1 \quad (64)$$

$$\dot{\epsilon}_\theta^{(p)} = \lambda_1 \frac{\partial f_1}{\partial \sigma_\theta} + \lambda_2 \frac{\partial f_2}{\partial \sigma_\theta} = \lambda_1 + \lambda_2 \quad (65)$$

$$\dot{\epsilon}_z^{(p)} = \lambda_1 \frac{\partial f_1}{\partial \sigma_z} + \lambda_2 \frac{\partial f_2}{\partial \sigma_z} = -N\lambda_2 \quad (66)$$

so that

$$[\epsilon_r^{(p)} + N\epsilon_\theta^{(p)} + \epsilon_z^{(p)}] = 0, \quad \epsilon_r^{(p)} + N\epsilon_\theta^{(p)} + \epsilon_z^{(p)} = g(r) \quad (67)$$

Let the plastic strains be $\bar{\epsilon}_r^{(p)}$, $\bar{\epsilon}_\theta^{(p)}$, $\bar{\epsilon}_z^{(p)}$ at r when the load p_b has placed the plastic-plastic interface of the middle and outer zones at r . Then, when $\bar{R} = r$,

$$g(r) = \bar{\epsilon}_r^{(p)} + N\bar{\epsilon}_\theta^{(p)} + \bar{\epsilon}_z^{(p)} \quad (68)$$

But in the outer plastic zone the plastic strain relations (40) hold so (68) reduces to $g(r) = 0$. Hence (67) becomes

$$\epsilon_r^{(p)} + N\epsilon_\theta^{(p)} + \epsilon_z^{(p)} = 0 \quad (69)$$

The plane strain condition is $\epsilon_z = \epsilon_z^{(e)} + \epsilon_z^{(p)} = 0$ which, when combined with (69) gives

$$\epsilon_r = -N\epsilon_\theta + N\epsilon_\theta^{(e)} + \epsilon_r^{(e)} + \epsilon_z^{(e)} \quad (70)$$

Stress relationships (57) and $\sigma_r = \sigma_z$ reduces Hooke's law (21), (22) and (23), to

$$E\epsilon_r^{(e)} = E\epsilon_z^{(e)} = [1 - (N+1)\nu]\sigma_r + \nu\sigma_u \quad (71)$$

$$E\epsilon_\theta^{(e)} = (N-2\nu)\sigma_r - \sigma_u \quad (72)$$

where σ_r is given by (60). Substitution of the elastic strains (71) and (72) in the radial strain expression (70) followed by substitution of the resulting ϵ_r in the compatibility equation (28) leads to the equation

$$2G \frac{d}{dr} (r^{N+1} \epsilon_\theta) = - \frac{N^2 + 2 - 2(2N+1)\nu}{(N-1)(1+\nu)} [(N-1)p_a + \sigma_u] \frac{r^{2N-1}}{a^{N-1}} + \frac{(N+2)(1-2\nu)\sigma_u}{(N-1)(1+\nu)} \cdot r^N \quad (73)$$

Integrating (73) from r to \bar{R} gives

$$\begin{aligned} 2G\epsilon_\theta = 2G\epsilon_\theta(\bar{R}) \cdot \left(\frac{\bar{R}}{r}\right)^{N+1} + \frac{N^2 + 2 - 2(2N+1)\nu}{2N(N-1)(1+\nu)} [(N-1)p_a + \sigma_u] \left[\left(\frac{\bar{R}}{r}\right)^{2N} - 1 \right] \left(\frac{r}{a}\right)^{N-1} \\ - \frac{(N+2)(1-2\nu)\sigma_u}{(N-1)(N+1)(1+\nu)} \left[\left(\frac{\bar{R}}{r}\right)^{N+1} - 1 \right] \end{aligned} \quad (74)$$

where $2G\epsilon_\theta(\bar{R})$ is determined by (48). From (70), (71) and (72) the radial strain is determined by

$$2G\epsilon_r = -2G\epsilon_\theta N + \frac{N^2 + 2 - 2(2N+1)\nu}{1+\nu} \sigma_r - \frac{N-2\nu}{1+\nu} \sigma_u \quad (75)$$

Increased loading—inner plastic zone

During the increase in loading ($p_b > \bar{p}_b$) the stress field in the inner plastic zone, $a < r < \bar{R}$ remains constant. Formula (14) gives σ_r . Formulae for σ_θ and σ_z then follow from the yield condition (7) and the result (24). The differential equation (29) for the plastic strain $\epsilon_\theta^{(p)}$ still holds with increased loading but now we integrate from r to \bar{R} instead of r to R . Instead of (30) we obtain

$$2G\epsilon_\theta^{(p)} = 2G\epsilon_\theta^{(p)}(\bar{R}) \left(\frac{\bar{R}}{r}\right)^{N+1} - \frac{(N+1)(1-\nu)}{2N} [(N-1)p_a + \sigma_u] \left(\frac{\bar{R}}{a}\right)^{N-1} \left[\left(\frac{\bar{R}}{r}\right)^{N+1} - \left(\frac{r}{\bar{R}}\right)^{N-1} \right] \quad (76)$$

The other plastic strain components are still given by (17). The elastic strain components remain the same because the stress field is unchanged, so the elastic strains are given by (20), (25) and (26). In (76), $\epsilon_\theta^{(p)}(\bar{R})$ is found by setting $r = \bar{R}$ in (74) to give $\epsilon_\theta(\bar{R})$, $r = \bar{R}$ in (72) to give $\epsilon_\theta^{(e)}(\bar{R})$ and applying strain summation (19); note that for (72) $\sigma_r(\bar{R})$ is given by (35).

Numerical example (Case 1, Table I)

A numerical example is presented to illustrate a comparison between the stress and strain fields obtained by the analysis and the NONSAP finite element code augmented by the

Mohr-Coulomb yield condition. The data are

Poisson's ratio	$\nu = 1/5$
Friction angle function	$N = 3(\varphi = 30^\circ)$
Unconfined crush strength	$\sigma_u = 2000 \text{ lb/in}^2$
Internal pressure	$p_a = 500 \text{ lb/in}^2$

We can load initially with $p_a = p_b$ until $p_a = 500 \text{ lb/in}^2$ without yielding because $2N\nu = 6/5 > 1$; see condition (4). The chosen numerical values result in $(N+1)\nu = 4/5 < 1$ and $\nu\sigma_u/[1 - (N+1)\nu] = 2000 \text{ lb/in}^2$, so that inequalities (9) are satisfied. Hence $p_a < 2\nu\bar{p}_b$ and the initial yield pressure is determined by formula (8), which gives $\bar{p}_b = 2000 \text{ lb/in}^2$ ($2\nu\bar{p}_b = 800 \text{ lb/in}^2$). The value of the far-field loading when $R = \bar{R}$, the radius of the elastic-plastic interface when $\sigma_r(R) = \sigma_z(R)$, is $\bar{p}_b = 5000 \text{ lb/in}^2$ according to formula (37); the stresses at $r = \bar{R}$ are $\sigma_r(\bar{R}) = \sigma_z(\bar{R}) = -2000 \text{ lb/in}^2$ and $\sigma_\theta(\bar{R}) = -8000 \text{ lb/in}^2$, according to formulae (35) and (36). Formula (34) gives $\bar{R} = \sqrt{2}a$. Another result that can readily be calculated is the circumferential strain at the hole when $p_b = \bar{p}_b$. According to formulae (26) and (30), $2G\varepsilon_\theta^{(e)}(a) = -2700 \text{ lb/in}^2$ and $2G\varepsilon_\theta^{(p)}(a) = -11,200 \text{ lb/in}^2$ so that $2G\varepsilon_\theta(a) = 13,900 \text{ lb/in}^2$. If, for simplicity, $G = 10^6 \text{ lb/in}^2$, $\varepsilon_\theta(a) = 0.70$ per cent when $p_b = \bar{p}_b = 5000 \text{ lb/in}^2$, that is, the tunnel diameter has been reduced by 0.70 per cent.

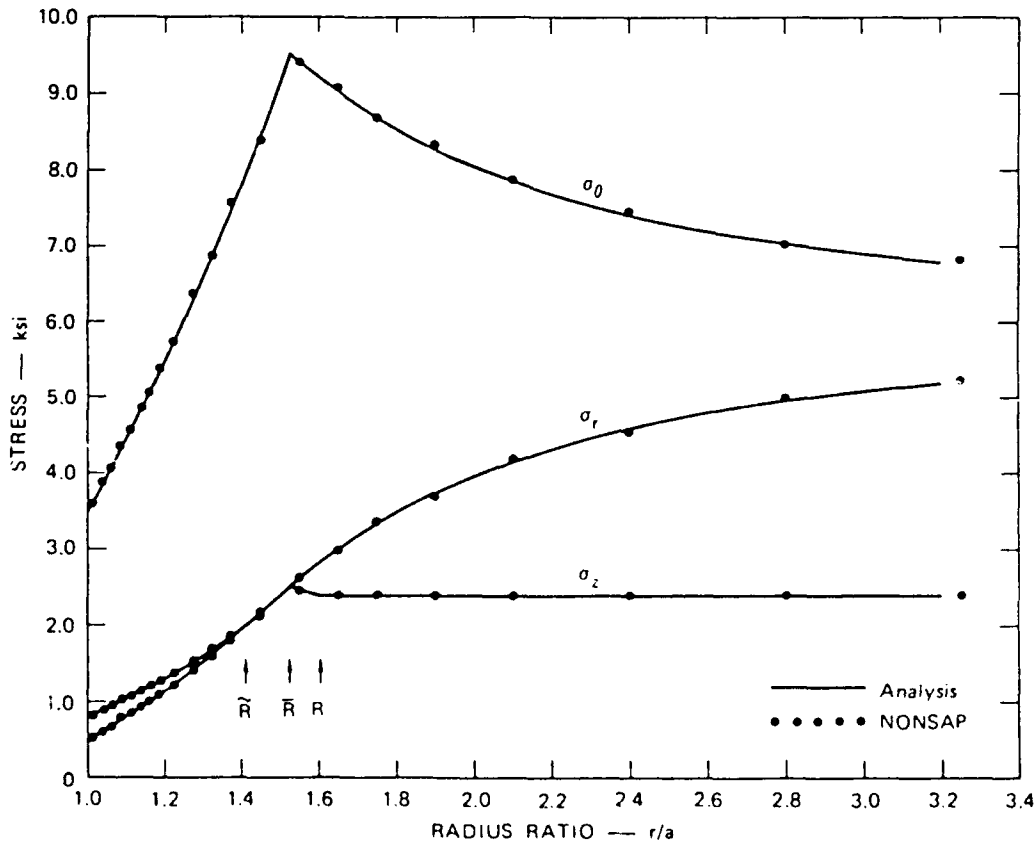


Figure 2. Comparison of analytical and NONSAP stress fields

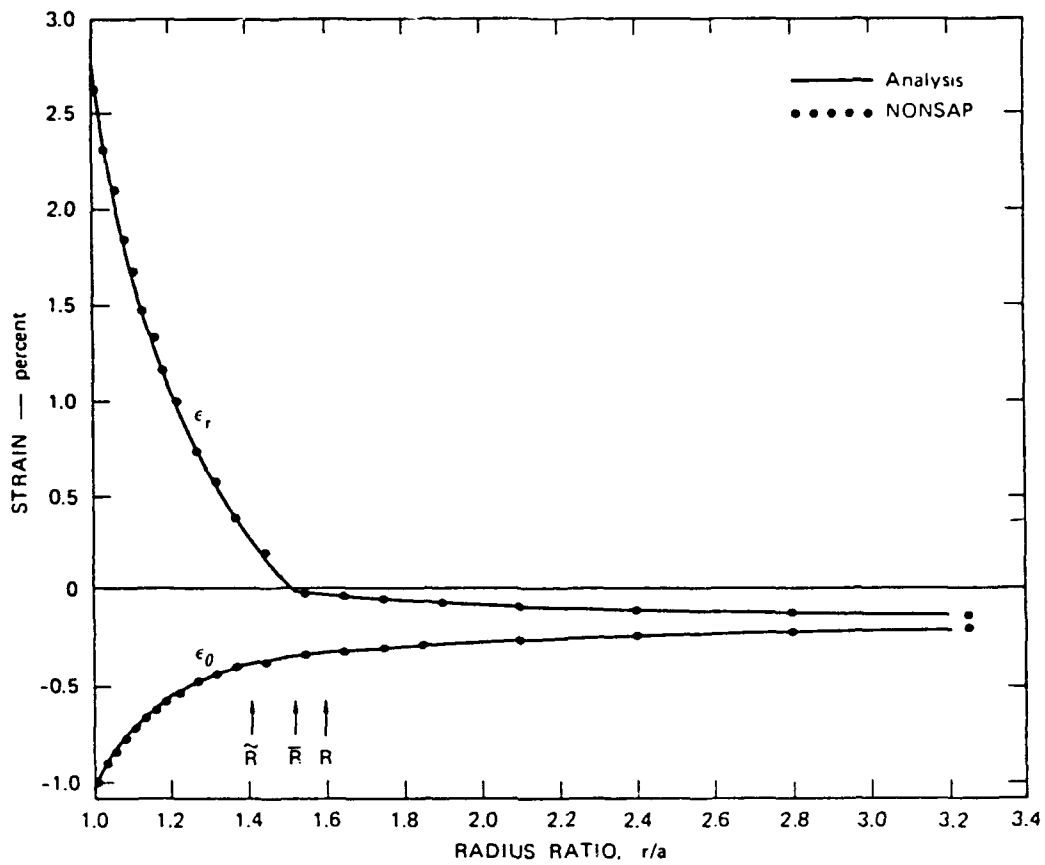


Figure 3. Comparison of analytical and NONSAP strain fields

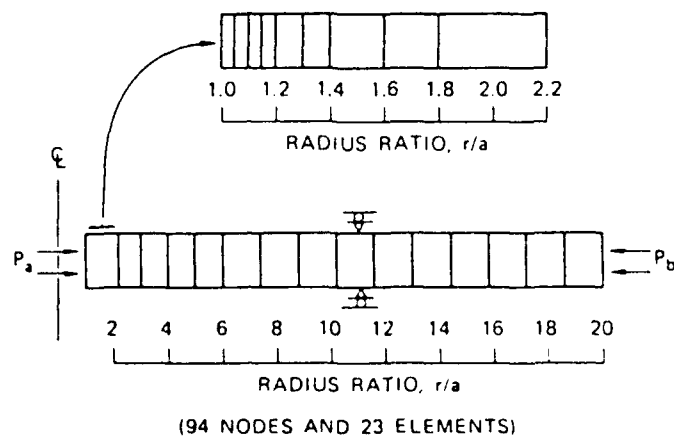


Figure 4. NONSAP axisymmetric finite element model

For loading above \bar{p}_b transcendental equations (56) and (61) have to be solved to find \bar{R} and R before stress and strain formulae can be evaluated. However, in many cases the constant α of formula (45) is approximately unity and rapid estimations can be made.

Figures 2 and 3 afford a comparison of the stress and strain fields for a far-field pressure of $p_b = 6000 \text{ lb/in}^2$; for the strain calculations the modulus of rigidity was taken as $G = 10^6 \text{ lb/in}^2$. Figure 4 shows the finite element model used in the NONSAP calculations. It is clear from the excellent agreement that for problems of this type the Mohr-Coulomb yield condition was correctly integrated into the NONSAP finite element code.

ACKNOWLEDGEMENTS

The authors benefited greatly from discussions with H. E. Lindberg and P. E. Senseny on the deformation of structures embedded in rock. The authors also appreciate the assistance of B. B. Bain for programming of the analytical solution.

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APPENDIX B

FLORENCE AND SCHWER CASE II SOLUTION

The following paper (Reference 4) was prepared under a DNA contract. It is reproduced here in its entirety with the concurrence of the senior author.

AXISYMMETRIC COMPRESSION OF A MOHR-COULOMB MEDIUM
AROUND A CIRCULAR HOLE

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SUMMARY

An analytical solution is presented for the stress and strain fields in a Mohr-Coulomb material in plane strain around a circular hole when it is compressed by an axisymmetric far-field pressure. Several solutions arise involving one to three plastic zones depending on the material properties and the loading. A solution involving three plastic zones was presented in reference 1. The solution presented here involves two plastic zones. The solution was used to further validate the functioning of the Mohr-Coulomb yield condition that was added to the NONSAP finite element code. Stress and strain field comparisons are made.

INTRODUCTION

The purpose of this paper is to provide an analytical solution of a rock mechanics problem that can be used to validate the predictions of finite element codes. The problem, which is of interest in itself, is the determination of the stress and strain fields surrounding a tunnel in a Mohr-Coulomb rock subjected far from the tunnel to an axisymmetric pressure. Plane strain conditions prevail and an internal pressure is allowed to develop at the hole boundary. Such a pressure may be considered as arising from a metal liner at yield or from a crushable back-packing material between the liner and the rock. To simplify the analysis and to provide a defined loading history this internal pressure is assumed to be equal to the far-field pressure as the far-field pressure is increased (loading) until the internal pressure reaches its final value, whereupon it remains constant with subsequent loading.

The rock behavior is elastic-perfectly plastic obeying the Mohr-Coulomb yield condition and associated flow rule; hence, dilatation is included. It will be seen that several cases can arise that depend on the relative magnitudes of the principal stresses and these relative magnitudes depend on the values of Poisson's ratio and the friction angle. One of these cases is treated in reference 1. It involves two adjacent faces and the common edge of the yield pyramid, and three corresponding plastic zones. The case treated here involves one face and an edge, and two corresponding plastic zones, as shown in Figure 1. In this work the finite-element code employed is NONSAP² to which the Mohr-Coulomb yield condition presented in reference 3 has been added. Comparisons are made of stress and strain fields obtained by analysis and code.

The motivation for the code validation analysis was to provide support for the experimental and theoretical investigations of Lindberg and Kennedy⁴ and Lindberg and Senseny⁵ into the response of buried cylindrical structures to quasi-static loading.

INITIAL LOADING. OUTER ELASTIC ZONE

Let p_a be the pressure acting on the hole boundary of radius a and let p_b be the pressure at infinity (far-field). If $p_a = p_b$ on initial loading, equilibrium requires that the radial and circumferential stress components are equal, that is, $\sigma_r = \sigma_\theta = -p_b$. If the material is elastic and in a plane strain condition, the axial stress is $\sigma_z = -2\nu p_b$ throughout, where ν is Poisson's ratio ($0 < \nu < 1/2$). Hence, we have the relationship $\sigma_\theta = \sigma_r < \sigma_z$. With increased loading yielding will occur if

$$f = \sigma_\theta - N\sigma_z + \sigma_u = 0 \quad (1)$$

where

$$N = \frac{1 + \sin\varphi}{1 - \sin\varphi} \quad (2)$$

and $\sigma_u > 0$ is the unconfined crush strength, φ being the friction angle. Substitution of the stress components in yield condition (1) gives

$$p_a = p_b = \frac{\sigma_u}{1 - 2N\nu} \quad (3)$$

For $p_b > 0$, the material will not yield if $2N\nu > 1$. However, if $2N\nu < 1$ the material will yield throughout when Eq. (3) is satisfied. We shall therefore consider design values of internal pressure p_a obeying the inequality

$$p_a < \frac{\sigma_u}{1 - 2N\nu} \quad \text{when } 2N\nu < 1 \quad (4)$$

and for p_a , so far, unrestricted when $2N\nu > 1$.

Once the design internal pressure has been reached it is held constant and the far-field pressure is increased. While the material remains elastic the stress and strain fields about a hole of radius $r = a$ are

$$\sigma_r = -p_b + (p_b - p_a) \frac{a^2}{r^2} \quad \sigma_\theta = -p_b - (p_b - p_a) \frac{a^2}{r^2} \quad \sigma_z = 2\nu p_b \quad (5)$$

$$2G\epsilon_r = - (1 - 2\nu)p_b + (p_b - p_a) \frac{a^2}{r^2} \quad 2G\epsilon_\theta = - (1 - 2\nu)p_b - (p_b - p_a) \frac{a^2}{r^2} \quad \epsilon_z = 0 \quad (6)$$

where, for our loading history, $p_a < p_b$. In (5), G is the shear modulus. We see immediately from (5) that $\sigma_\theta < \sigma_r$ and $\sigma_\theta < \sigma_z$ throughout. Also,

$\sigma_r(\infty) < \sigma_z$, but at the hole the inequalities are conditional, that is, $\sigma_r(a) < \sigma_z$ if $p_a > 2\nu p_b$ and $\sigma_z < \sigma_r(a)$ if $p_a < 2\nu p_b$. It can readily be shown that with increasing loading yielding occurs first at the hole.

We now examine the effect these inequalities have on the initial yielding at the hole.

INITIAL YIELDING

Let yielding occur at $r = a$ when $p_b = \bar{p}_b$. If $p_a < 2\nu \bar{p}_b$ when yielding occurs, then $\sigma_\theta < \sigma_z < \sigma_r$ at $r = a$ and the yield condition is

$$f = \sigma_\theta - N\sigma_r + \sigma_u = 0 \quad (7)$$

Substitution of $\sigma_\theta(a) = -2\bar{p}_b + p_a$ and $\sigma_r(a) = -p_a$, from (5), in Eq. (7) leads to the yield pressure

$$\bar{p}_b = \frac{1}{2}[(N+1)p_a + \sigma_u] \quad (8)$$

Satisfaction of the inequality $p_a < 2\nu \bar{p}_b$ now requires

$$p_a < \frac{\sigma_u}{1 - (N+1)\nu} \quad (N+1)\nu < 1 \quad (9)$$

but places no restriction on positive p_a when $(N+1)\nu > 1$

If $2\nu\bar{p}_b < p_a < \bar{p}_b$ when yielding occurs, then $\sigma_\theta < \sigma_r < \sigma_z$ at $r = a$ and the yield condition is Eq. (1). Substitution of $\sigma_\theta(a) = -2\bar{p}_b + p_a$ and $\sigma_z(a) = -2\nu\bar{p}_b$ in Eq. (1) leads to the yield pressure

$$\bar{p}_b = \frac{p_a + \sigma_u}{2(1 - N\nu)} \quad N\nu < 1 \quad (10)$$

Yield condition (1) does not govern yielding if $N\nu > 1$. In this case, p_b increases until $p_a < 2\nu\bar{p}_b$ so that yielding is governed by condition (7) and \bar{p}_b is given by formula (8); because $(N + 1)\nu > 1$, no restriction is placed on p_a . For $N\nu < (N + 1)\nu < 1$, inequality $2\nu\bar{p}_b < p_a < \bar{p}_b$ and the yield pressure (10) restrict p_a to the ranges

$$\frac{\nu\sigma_u}{1 - (N + 1)\nu} < p_a < \frac{\sigma_u}{1 - 2N\nu} \quad 0 < N\nu < \frac{1}{2} \quad (11)$$

$$\frac{\nu\sigma_u}{1 - (N + 1)\nu} < p_a \quad \frac{1}{2} < N\nu < (N + 1)\nu < 1 \quad (12)$$

The analysis of case 1 in Table 1 is contained in reference 1. Here, we present the analysis of case 2 for $\bar{p}_b < p_b$, where \bar{p}_b is given by (10).

INITIAL GROWTH OF PLASTIC ZONE

As the far-field pressure is increased beyond the initial yield value \bar{p}_b an annular plastic zone of outer radius $r = R$ expands from the hole of radius $r = a$. Within this zone yielding is governed by condition (1) and the associated flow rule

$$\dot{\epsilon}_r^{(p)} = \lambda \frac{\partial f}{\partial \sigma_r} = 0 \quad \dot{\epsilon}_\theta^{(p)} = \lambda \frac{\partial f}{\partial \sigma_\theta} = \lambda \quad \dot{\epsilon}_z^{(p)} = \lambda \frac{\partial f}{\partial \sigma_z} = -N\lambda \quad (13)$$

where the dots may be interpreted as differentiation with respect to the loading p_b . From flow rule (13) it follows that

Table 1

INITIAL YIELDING CONDITIONS

Property Relations	Internal Pressure	Yield Pressure	Stress Order	Yielding Case
$N\nu < (N+1)\nu < 1$	$p_a < \frac{\nu\sigma_u}{1 - (N+1)\nu}$	$\bar{p}_b = \frac{1}{2}[(N+1)p_a + \sigma_u]$	$\sigma_\theta < \sigma_z < \sigma_r$	(1) $p_a < 2\nu\bar{p}_b$
$1 < N\nu < (N+1)\nu$ $N\nu < 1 < (N+1)\nu$	$p_a > 0$			
$\frac{1}{2} < N\nu < (N+1)\nu < 1$	$p_a > \frac{\nu\sigma_u}{1 - (N+1)\nu}$	$\bar{p}_b = \frac{p_a + \sigma_u}{2(1 - N\nu)}$	$\sigma_\theta < \sigma_r < \sigma_z$	(2) $2\nu\bar{p}_b < p_a < \bar{p}_b$
$0 < N\nu < \frac{1}{2}$	$\frac{\nu\sigma_u}{1 - (N+1)\nu} < p_a < \frac{\sigma_u}{1 - 2N\nu}$			

$$N\epsilon_{\theta}^{(p)} + \epsilon_z^{(p)} = g_1(r) \quad \epsilon_r^{(p)} = g_2(r) \quad (14)$$

which means that the strain expressions in (14) evaluated at a fixed radius r remain constant as loading proceeds. Initially, when the elastic-plastic radius R is at this radius r , the plastic strains are zero, so that $g_1(r) = 0$ and $g_2(r) = 0$. Hence the strain expressions in (14) become

$$N\epsilon_{\theta}^{(p)} + \epsilon_z^{(p)} = 0 \quad \epsilon_r^{(p)} = 0 \quad (15)$$

Strain increments are taken as the sum of the elastic and plastic strain increments, that is,

$$\dot{\epsilon}_r = \dot{\epsilon}_r^{(e)} + \dot{\epsilon}_r^{(p)} \quad \dot{\epsilon}_{\theta} = \dot{\epsilon}_{\theta}^{(e)} + \dot{\epsilon}_{\theta}^{(p)} \quad \dot{\epsilon}_z = \dot{\epsilon}_z^{(e)} + \dot{\epsilon}_z^{(p)} \quad (16)$$

Now $\epsilon_r = (\epsilon_r^{(e)} + \epsilon_r^{(p)}) = g(r)$, and when $R = r$ we have $\epsilon_r = \epsilon_r^{(e)}$ and $\epsilon_r^{(p)} = 0$, so that $g(r) = 0$. Using the same argument for the other two strain components leads to

$$\epsilon_r = \epsilon_r^{(e)} + \epsilon_r^{(p)} \quad \epsilon_{\theta} = \epsilon_{\theta}^{(e)} + \epsilon_{\theta}^{(p)} \quad \epsilon_z = \epsilon_z^{(e)} + \epsilon_z^{(p)} \quad (17)$$

Plane strain requires $\epsilon_z = 0$, so (15) and (17) give the plastic and total strains as

$$\epsilon_r^{(p)} = 0 \quad N\epsilon_{\theta}^{(p)} = \epsilon_z^{(e)} \quad \epsilon_z^{(p)} = -\epsilon_z^{(e)}$$

and

$$\epsilon_r = \epsilon_r^{(e)} \quad \epsilon_{\theta} = \epsilon_{\theta}^{(e)} + \frac{1}{N} \epsilon_z^{(e)} \quad \epsilon_z = 0 \quad (18)$$

Hooke's law relating the elastic strains to the stresses is

$$E \epsilon_r^{(e)} = \sigma_r - \nu(\sigma_\theta + \sigma_z) \quad (19)$$

$$E \epsilon_\theta^{(e)} = \sigma_\theta - \nu(\sigma_z + \sigma_r) \quad (20)$$

$$E \epsilon_z^{(e)} = \sigma_z - \nu(\sigma_r + \sigma_\theta) \quad (21)$$

where E is Young's modulus. Elimination of σ_z from (19), (20), and (21) by using the yield condition (1), and substitution of the resulting elastic strains in (18) gives the stress-strain relationships

$$[N^2 + 1 - (N + 1)^2 \nu] \sigma_r = (N^2 + 1 - 2N\nu) 2G\epsilon_r + N(N + 1) \nu 2G\epsilon_\theta + (N - 1) \nu \sigma_u \quad (22)$$

$$[N^2 + 1 - (N + 1)^2 \nu] \sigma_\theta = N(N + 1) \nu 2G\epsilon_r + N^2 2G\epsilon_\theta - [1 - (N + 1) \nu] \sigma_u \quad (23)$$

Substitution of these stresses in the equation of equilibrium

$$r \frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = 0 \quad (24)$$

and substitution of the strain-displacement relations

$$\epsilon_r = \frac{du}{dr} \quad \epsilon_\theta = \frac{u}{r} \quad (25)$$

leads to the radial displacement equation

$$r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} - \alpha^2 u = - \frac{\alpha^2}{N} \frac{(1 - 2\nu) \sigma_u}{2G} \quad (26)$$

where

$$\alpha^2 = N^2 / (N^2 + 1 - 2N\nu) \quad (27)$$

The displacement solution of (26) and the corresponding strains are

$$2Gu = A_1 r^\alpha + A_2 r^{-\alpha} + Ar \quad (28)$$

$$2G\epsilon_r = \alpha A_1 r^{\alpha-1} - \alpha A_2 r^{-\alpha-1} + A \quad (29)$$

$$2G\epsilon_\theta = A_1 r^{\alpha-1} + A_2 r^{-\alpha-1} + A \quad (30)$$

where

$$A = - \frac{(1 - 2\nu)\sigma_u}{1 - 2N\nu} \quad (31)$$

In the elastic zones the stresses and strains are given by (5) and (6) with $a = R$ and $p_a = -\sigma_r(R)$. Thus, yield condition (1) applied at the elastic-plastic interface $r = R$ gives

$$\sigma_r(R) = - 2(1 - N\nu)p_b + \sigma_u \quad (32)$$

$$2G\epsilon_\theta(R) = - 2(N - 1)\nu p_b - \sigma_u \quad (33)$$

Continuity of displacement at $r = R$ is assured by equating (33) to (30) and continuity of radial stress at $r = R$ is assured by equating (32) to the stress in (22) after eliminating the strains with (29) and (30). These operations provide two algebraic equations for A_1 and A_2 with the solution

$$A_1 = B_1 R^{1-\alpha} \frac{\alpha[\sigma_u - (1 - 2N\nu)p_b]}{N^2(1 - 2N\nu)} \quad (34)$$

$$A_2 = B_2 R^{1+\alpha} \frac{\alpha[\sigma_u - (1 - 2N\nu)p_b]}{N^2(1 - 2N\nu)} \quad (35)$$

where

$$B_1 = \left[\frac{N}{\alpha} - (N+1)\nu \right] (N-1)N\nu + \left[N^2 + 1 - (N+1)^2\nu \right] (1-N\nu) \quad (36)$$

$$B_2 = \left[\frac{N}{\alpha} + (N+1)\nu \right] (N-1)N\nu - \left[N^2 + 1 - (N+1)^2\nu \right] (1-N\nu) \quad (37)$$

The radius R of the elastic-plastic interface is found by satisfying the pressure boundary condition at the hole. Hence, (22) with $\sigma_r = -p_a$, and with the strains (29) and (30) at $r = a$ having A_1 and A_2 given by (34) and (35), provides the equation

$$\begin{aligned} & \left[\frac{N}{\alpha} + (N+1)\nu \right] B_1 \left(\frac{R}{a} \right)^{1-\alpha} - \left[\frac{N}{\alpha} - (N+1)\nu \right] B_2 \left(\frac{R}{a} \right)^{1-\alpha} \\ & = \frac{N}{\alpha} \left[N^2 + 1 - (N+1)^2\nu \right] \frac{\sigma_u - (1-2N\nu)p_a}{\sigma_u - (1-2N\nu)p_b} \end{aligned} \quad (38)$$

for the radius R , B_1 and B_2 being given by (36) and (37).

The above description of yielding applies with continued loading as long as $\sigma_r < \sigma_z$ in the plastic zone. A numerical study of the stress difference $\sigma_z - \sigma_r$ was performed for values of N and ν that satisfied the inequality $N\nu < (N+1)\nu < 1$ and it was found that $\sigma_z - \sigma_r$ decreases as p_b increases and becomes zero first at the hole, $r = a$. Explicit expressions describing the stress field are

$$C\sigma_r = N \left[\frac{N}{\alpha} + (N+1)\nu \right] \frac{A_1}{r^{1-\alpha}} - N \left[\frac{N}{\alpha} - (N+1)\nu \right] \frac{A_2}{r^{1+\alpha}} - \frac{C\sigma_u}{1-2N\nu} \quad (39)$$

$$C\sigma_{\theta} = \alpha N \left[\frac{N}{\alpha} + (N+1)\nu \right] \frac{A_1}{r^{1-\alpha}} + \alpha N \left[\frac{N}{\alpha} - (N+1)\nu \right] \frac{A_2}{r^{1+\alpha}} - \frac{C\sigma_u}{1-2N\nu} \quad (40)$$

$$C\sigma_z = \alpha \left[\frac{N}{\alpha} + (N+1)\nu \right] \frac{A_1}{r^{1-\alpha}} + \alpha \left[\frac{N}{\alpha} - (N+1)\nu \right] \frac{A_2}{r^{1+\alpha}} - \frac{C2\nu\sigma_u}{1-2N\nu} \quad (41)$$

where

$$C = N^2 + 1 - (N+1)^2\nu \quad (42)$$

If we set $\sigma_r(a) = \sigma_z(a) = -p_a$ in (39) and (41) we obtain two equations for the loading p'_b and the corresponding elastic-plastic radius R' that may be put in the form

$$\left(\frac{R'}{a} \right)^{2\alpha} = \frac{B_1 \left[\frac{N}{\alpha} + (N+1)\nu \right] \left[\left(\frac{2N\nu}{\alpha} - 1 \right) \sigma_u - \left(\frac{N}{\alpha} - 1 \right) (1 - 2N\nu) p_a \right]}{B_2 \left[\frac{N}{\alpha} - (N+1)\nu \right] \left[\left(\frac{2N\nu}{\alpha} + 1 \right) \sigma_u - \left(\frac{N}{\alpha} + 1 \right) (1 - 2N\nu) p_a \right]} \quad (43)$$

$$\left[\sigma_u - (1 - 2N\nu) p'_b \right]^2 =$$

$$\frac{C^2 N^2 \left[\left(\frac{2N\nu}{\alpha} + 1 \right) \sigma_u - \left(\frac{N}{\alpha} + 1 \right) (1 - 2N\nu) p_a \right] \left[\left(\frac{2N\nu}{\alpha} - 1 \right) \sigma_u - \left(\frac{N}{\alpha} - 1 \right) (1 - 2N\nu) p_a \right]}{4B_1 B_2 \alpha^2 \left[\frac{N}{\alpha} - (N+1)\nu \right] \left[\frac{N}{\alpha} + (N+1)\nu \right] \left(\frac{R'}{a} \right)^2} \quad (44)$$

INCREASED LOADING

It is postulated that with increased loading ($p_b > p'_b$) a radius $\bar{R} > a$ exists where $\sigma_r(\bar{R}) = \sigma_z(\bar{R})$. In the outer annular plastic zone $\bar{R} < r < R$ we still have $\sigma_{\theta} < \sigma_r < \sigma_z$ (Figure 1) and the stress and strain fields are given by (39) - (42) and (29) - (31) but the radius $R > R'$ is no longer given by (38). In fact, satisfying the condition $\sigma_r(\bar{R}) = \sigma_z(\bar{R})$

by equating (39) and (41) with $r = \bar{R}$ leads to

$$\left(\frac{N}{\alpha} - 1\right) \left[\frac{N}{\alpha} + (N+1)\nu\right] B_1 \left(\frac{R}{\bar{R}}\right)^{1-\alpha} - \left(\frac{N}{\alpha} + 1\right) \left[\frac{N}{\alpha} - (N+1)\nu\right] B_2 \left(\frac{R}{\bar{R}}\right)^{1+\alpha} \\ = \frac{CN^2(1-2\nu)\sigma_u}{\alpha^2 \left[\sigma_u - (1-2N\nu)p_b\right]} \quad (45)$$

for the radius ratio R/\bar{R} . The individual values depend on the fields in the zone $a < r < \bar{R}$.

It is also postulated that with increased loading $\sigma_r = \sigma_z$ in the plastic zone $a < r < \bar{R}$, as shown in Figure 1, so that yielding is governed by

$$f_1 = \sigma_\theta - N\sigma_r + \sigma_u = 0 \quad (46)$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \quad (47)$$

Equation (46) allows the elimination of σ_θ from the equilibrium equation (24) to leave a differential equation with the solution

$$\sigma_r = - \left[p_a + \frac{\sigma_u}{N-1} \right] \left(\frac{r}{a} \right)^{N-1} + \frac{\sigma_u}{N-1} \quad (48)$$

which satisfies the pressure boundary condition at the hole. Continuity of the radial stress at the interface $r = \bar{R}$ is assured by setting $r = \bar{R}$ in (48) and (39) and equating the resulting expressions for $\sigma_r(\bar{R})$. This operation leads to the following equation for \bar{R} in terms of the ratio R/\bar{R} , which is the solution of (45),

$$\begin{aligned}
& \left[(N-1)p_a + \sigma_u \right] \left(\frac{\bar{R}}{a} \right)^{N-1} = \frac{N(1-2\nu)\sigma_u}{1-2N\nu} \\
& - \left[\left\{ \frac{N}{\alpha} + (N+1)\nu \right\} B_1 \left(\frac{R}{\bar{R}} \right)^{1-\alpha} - \left\{ \frac{N}{\alpha} - (N+1)\nu \right\} B_2 \left(\frac{R}{\bar{R}} \right)^{1+\alpha} \right] \frac{\alpha(N-1)[\sigma_u - (1-2N\nu)p_b]}{CN(1-2N\nu)}
\end{aligned} \tag{49}$$

The flow rule associated with the yield condition given by (46) and (47) is

$$\dot{\epsilon}_r^{(p)} = \lambda_1 \frac{\partial f_1}{\partial \sigma_r} + \lambda_2 \frac{\partial f_2}{\partial \sigma_r} = -N\lambda_1 \tag{50}$$

$$\dot{\epsilon}_\theta^{(p)} = \lambda_1 \frac{\partial f_1}{\partial \sigma_\theta} + \lambda_2 \frac{\partial f_2}{\partial \sigma_\theta} = \lambda_1 + \lambda_2 \tag{51}$$

$$\dot{\epsilon}_z^{(p)} = \lambda_1 \frac{\partial f_1}{\partial \sigma_z} + \lambda_2 \frac{\partial f_2}{\partial \sigma_z} = -N\lambda_2 \tag{52}$$

so that

$$\left[\epsilon_r^{(p)} + N\epsilon_\theta^{(p)} + \epsilon_z^{(p)} \right]' = 0 \quad \epsilon_r^{(p)} + N\epsilon_\theta^{(p)} + \epsilon_z^{(p)} = g(r) \tag{53}$$

Let the plastic strains be $\bar{\epsilon}_r^{(p)}$, $\bar{\epsilon}_\theta^{(p)}$, $\bar{\epsilon}_z^{(p)}$ at radius r when the load p_b has placed the plastic-plastic interface at r . Then, when $\bar{R} = r$

$$g(r) = \bar{\epsilon}_r^{(p)} + N\bar{\epsilon}_\theta^{(p)} + \bar{\epsilon}_z^{(p)} \tag{54}$$

But in the outer plastic zone the plastic strain relationships (15) hold so (54) reduces to $g(r) = 0$. Hence 53) becomes

$$\epsilon_r^{(p)} + N\epsilon_\theta^{(p)} + \epsilon_z^{(p)} = 0 \tag{55}$$

The plane strain condition is $\epsilon_z = \epsilon_z^{(e)} + \epsilon_z^{(p)} = 0$ which, when combined with (55) gives

$$\epsilon_r = -N\epsilon_\theta + N\epsilon_\theta^{(e)} + \epsilon_r^{(e)} + \epsilon_z^{(e)} \quad (56)$$

The yield condition (46) and $\sigma_z = \sigma_r$ reduces Hooke's law (19), (20), and (21) to

$$E\epsilon_r^{(e)} = E\epsilon_z^{(e)} = [1 - (N + 1)\nu]\sigma_r + \nu\sigma_u \quad (57)$$

$$E\epsilon_\theta^{(e)} = (N - 2\nu)\sigma_r - \sigma_u \quad (58)$$

where σ_r is given by (48). Substitution of the elastic strains (57) and (58) in the radial strain expression (56) followed by substitution of the resulting ϵ_r in the compatibility equation

$$r \frac{d\epsilon_\theta}{dr} + \epsilon_\theta - \epsilon_r = 0 \quad (59)$$

leads to the equation

$$\begin{aligned} 2G \frac{d}{dr} \left(r^{N+1} \epsilon_\theta \right) = & - \frac{N^2 + 2 - 2(2N + 1)\nu}{(N - 1)(1 + \nu)} [(N - 1)p_a + \sigma_u] \frac{r^{2N-1}}{a^{N-1}} \\ & + \frac{(N + 2)(1 - 2\nu)\sigma_u}{(N - 1)(1 + \nu)} \cdot r^N \end{aligned} \quad (60)$$

Integrating (60) from r to \bar{R} gives

$$\begin{aligned} 2G\epsilon_\theta = 2G\epsilon_\theta(\bar{R}) \cdot \left(\frac{\bar{R}}{r} \right)^{N+1} \\ + \frac{N^2 + 2 - 2(2N + 1)\nu}{2N(N - 1)(1 + \nu)} [(N - 1)p_a + \sigma_u] \left[\left(\frac{\bar{R}}{r} \right)^{2N} - 1 \right] \left(\frac{r}{a} \right)^{N-1} \\ - \frac{(N + 2)(1 - 2\nu)\sigma_u}{(N - 1)(N + 1)(1 + \nu)} \left[\left(\frac{\bar{R}}{r} \right)^{N+1} - 1 \right] \end{aligned} \quad (61)$$

where $2G\epsilon_{\theta}(\bar{R})$ is determined by (30). Explicitly,

$$2G\epsilon_{\theta}(R) = \left[B_1 \left(\frac{R}{\bar{R}} \right)^{1+\alpha} + B_2 \left(\frac{R}{\bar{R}} \right)^{1+\alpha} \right] \frac{\alpha [\sigma_u - (1 - 2\nu)p_b]}{N^2(1 - 2\nu)} - \frac{(1 - 2\nu)\sigma_u}{1 - 2\nu} \quad (62)$$

Substitution in (56) of the elastic strains (57) and (58), with σ_r given by (48), results in the radial strain formula

$$2G\epsilon_r = -2G\epsilon_{\theta} - \frac{N^2 + 2 - 2(2N + 1)\nu}{(N - 1)(1 + \nu)} [(N - 1)p_a + \sigma_u] \left(\frac{r}{a} \right)^{N-1} + \frac{(N + 2)(1 - 2\nu)\sigma_u}{(N - 1)(1 + \nu)}$$

During the analysis we postulated that a third plastic zone could exist next to the hole in which $\sigma_{\theta} < \sigma_z < \sigma_r$, as in case 1 of Table 1.¹ The result of the analysis showed that $\sigma_z = \sigma_r$ in the postulated third zone so that only two plastic zones exist in case 2.

NUMERICAL EXAMPLE

A numerical example is presented to illustrate a comparison between the stress and strain fields obtained by the analysis and the NONSAP finite element code augmented by the Mohr-Coulomb yield condition. The data are

Poisson's ratio	$\nu = 1/5$
Friction angle function	$N = 2 \quad (\varphi = 19.5^\circ)$
Unconfined crush strength	$\sigma_u = 3000 \text{ psi}$
Internal pressure	$p_a = 2000 \text{ psi}$

The values of Poisson's ratio and friction angle are such that $2N\nu = 4/5 < 1$ so the yielding with $p_a = p_b$ does not occur until

$p_a = p_b = 15,000$ psi, according to (3). Hence we can load with equal p_a and p_b until $p_a = p_b = 2000$ psi. Inequality (11) is $1500 \text{ psi} < p_a < 15,000 \text{ psi}$. If we now fix p_a and increase p_b yielding occurs when p_b reaches $\bar{p}_b = 4167$ psi, according to (10). As loading is increased a single plastic zone spreads until $p_b = p'_b = 4529$ psi and $R'/a = 1.10$, according to (43) and (44). Further loading creates two plastic zones and when $p_b = 6000$ psi, we obtain $R/\bar{R} = 1.31$ from (45), $\bar{R}/a = 1.19$ from (45) and (49), and hence $R/a = 1.55$. The tunnel closure at this loading is 0.86%, according to (61).

In many cases the constant α of formula (27) is approximately unity. This value allows all unknowns to be determined explicitly so that rapid estimations can be made.

Figures 2 and 3 provide a comparison of the stress and strain fields as obtained by the analysis and by the NONSAP finite-element code. Figure 4 shows the finite element model used in the NONSAP calculations. The far-field pressure is $p_b = 6000$ psi and a modulus of rigidity of $G = 10^6$ psi was taken for the strain calculations. It is clear from the excellent agreement that for problems of this type the Mohr-Coulomb yield condition was correctly integrated into the NONSAP finite-element code. The same conclusion was reached in reference 1 for case 1 of Table 1.

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Acknowledgements

The authors benefited greatly from discussions with H. E. Lindberg and P. E. Senseny on the deformation of structures embedded in rock. The authors also appreciate the assistance of B. B. Bain for programming of the analytical solution.

KEY WORDS

Mohr-Coulomb Medium

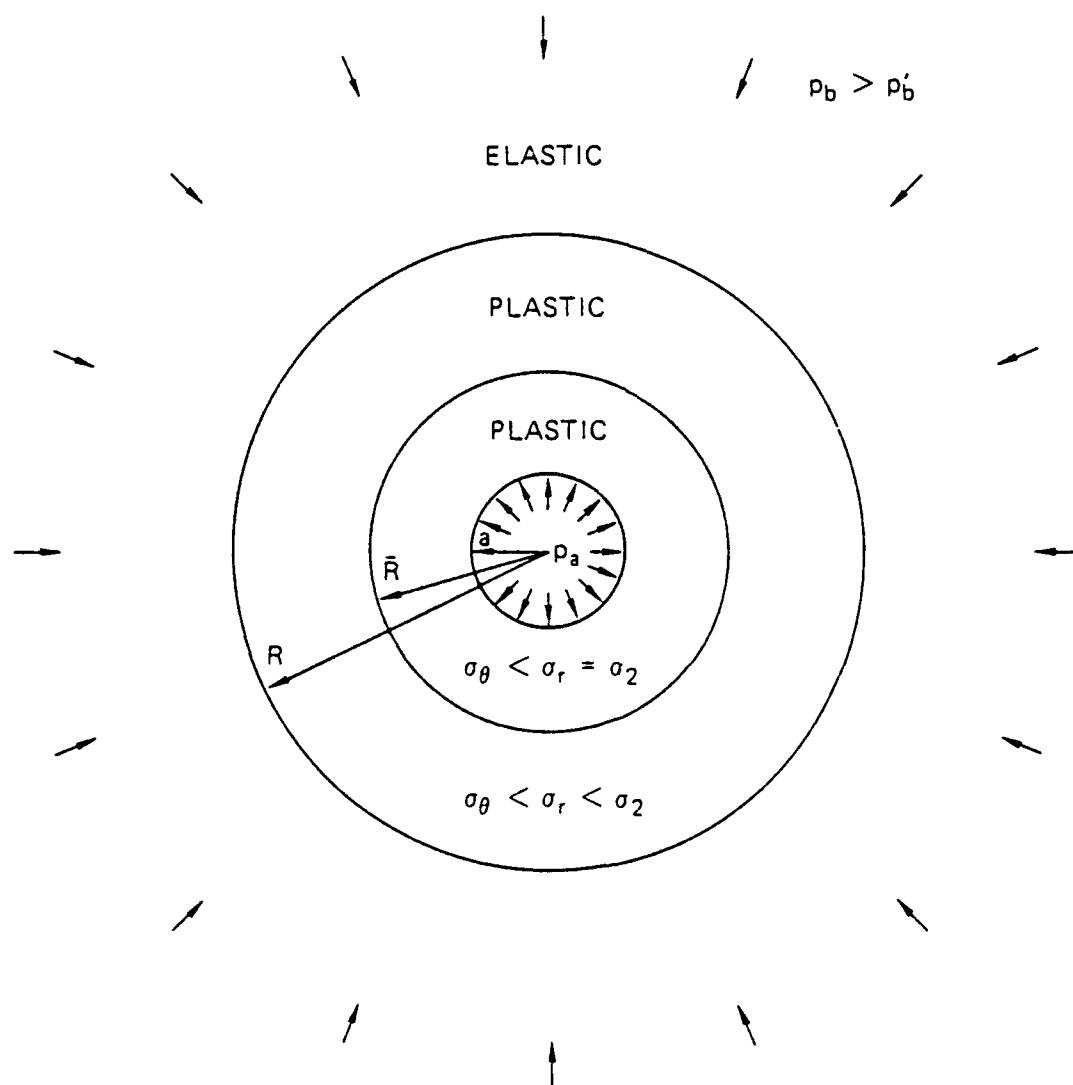
Plasticity

Axisymmetric Finite Element

NONSAP

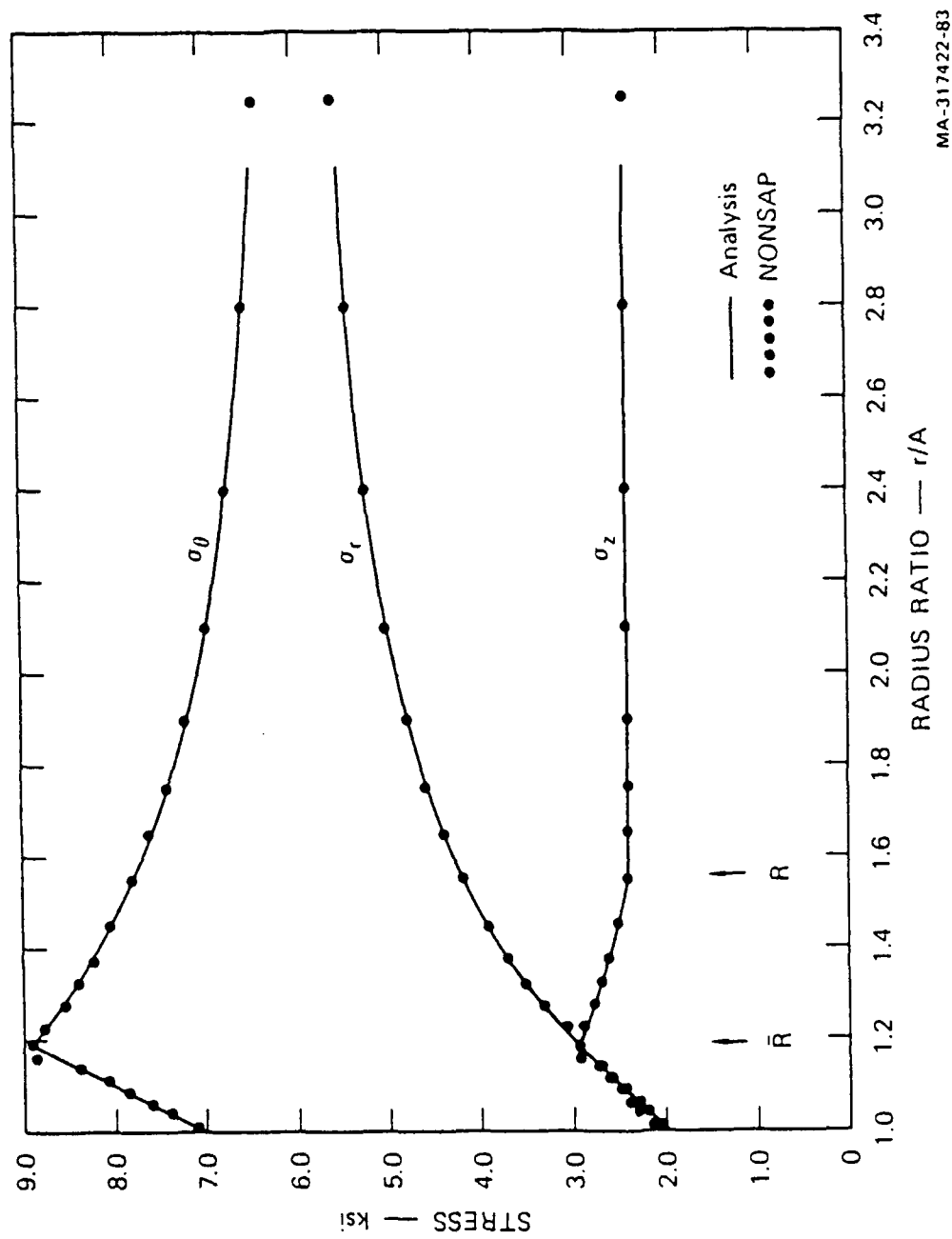
ILLUSTRATIONS

- 1 Outer Elastic and Three Plastic Zones
- 2 Comparison of Analytical and NONSAP Stress Fields
- 3 Comparison of Analytical and NONSAP Strain Fields
- 4 NONSAP Axisymmetric Finite Element Model

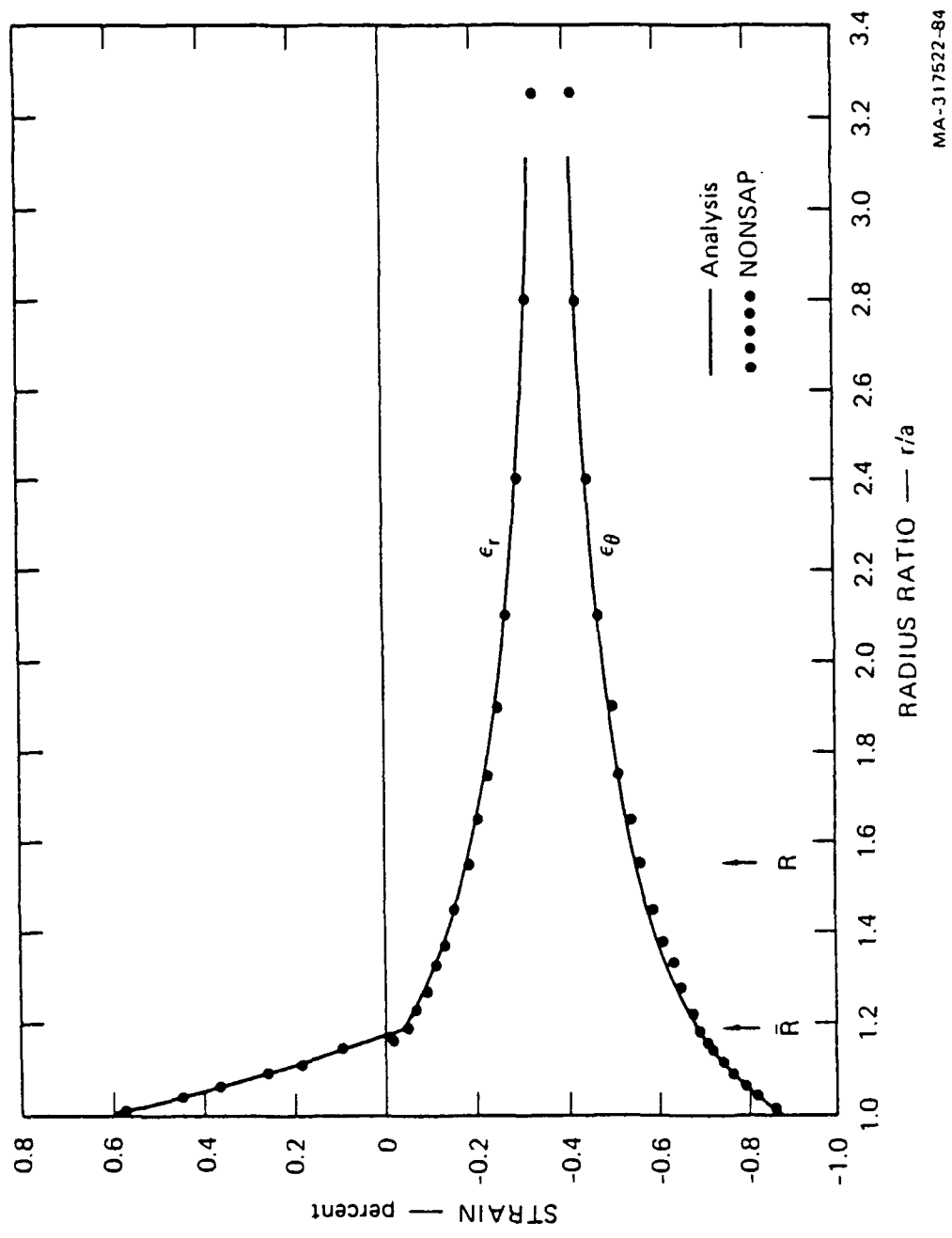


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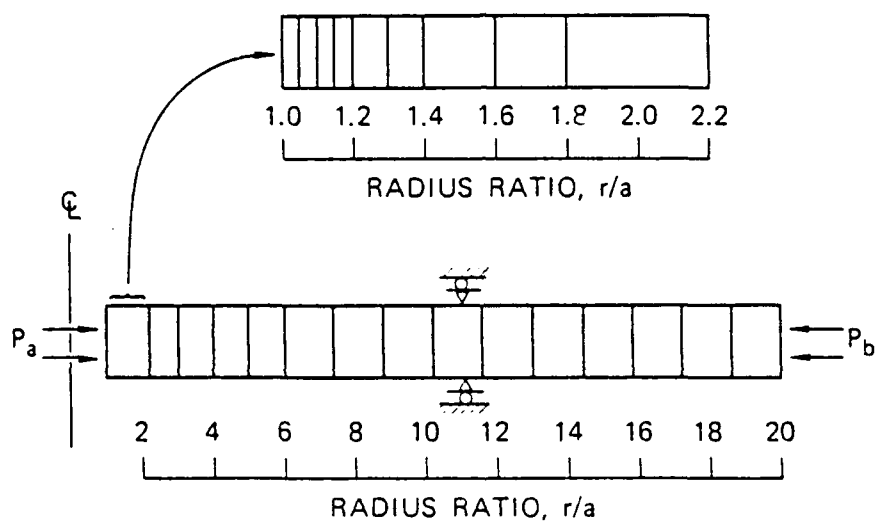
FIGURE 1 OUTER ELASTIC AND TWO PLASTIC ZONES



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(94 NODES AND 23 ELEMENTS)

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FIGURE 4

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